Chapter 6

Graphs

“The origins of graph theory are humble, even frivolous.”
N. Biggs, E. K. Lloyd, and R. J. Wilson
(Graph Theory: 1736-1936)

Graphs are ubiquitous in mathematics. They are used for defining the connectivity of networks and
for diagramming and understanding relationships of many kind. The famous “six degrees of separation”
problem takes place on a graph describing interpersonal links. Graphs are used to store contacts that
may spread a disease in epidemiology, data transmission links in the internet, or patterns of influence
between genes. The formalisms of graphs have partner formalisms in linear algebra and we will explore those
links in the later parts of this chapter.

6.1 Basic Definitions

A graph $G$ is composed of a set $V$ of vertices and a
set $E$ of edges with $E$ being a subset of the set of un-
ordered pairs from $V$. To compactly specify a graph
and its sets of vertices and edges we often use the
notation $G(V, E)$. We call a graph finite or infinite
depending on the size of its vertex set.

Definition 6.1 Suppose that $G(V, E)$ is a graph.
Then for $e = \{a, b\} \in E$ we call $a$ and $b$ the ends
of $e$. We say that $e$ is incident upon or incident
with $a$ and $b$.

Definition 6.2 Two vertices of a graph are adja-
cent if and only if they are the ends of an edge.

The edges of a graph are sometimes called the adja-
cency relation of the graph. In Problem 6.2 we check
to see if this is a good name. Recall that a relation
is a set of ordered pairs while edges of a graph are unordered pairs.

Definition 6.3 For a graph $G(V, E)$ the number of
edges incident on $v \in V$ is called the degree of $v$ in
$G$. This is denoted $\delta(v)$.

Notice that in a graph $G(V, E)$, we defined $E$ to be a
set. If $E$ is a multiset, a collection of edges in which
a given edge may appear more than once, we call the
number of appearances of an edge its multiplicity and
call that edge a multiple edge. We will, for the most
part, avoid multiple edges.

Definition 6.4 An edge of the form $\{a, a\}$ is called
a loop.

Definition 6.5 A graph with no loops or multiple
edges is called a simple graph.

Lemma 6.1 The number of vertices in a finite sim-
ple graph of odd degree must be even.

Proof: this proof is an exercise.

Definition 6.6 A list of the degrees of a finite graph
in descending numerical order is called the degree
sequence of a graph. Multiple equal entries may be
described in a shorthand fashion by using exponents
as in Example 6.1.

Example 6.1 Let $G(V, E)$ have vertex set

$$V = \{A, B, C, D, E, F, G\}$$

and edge set

$$E = \{\{A, B\}, \{A, E\}, \{A, F\}, \{B, C\}, \{B, F\}, \{C, D\},$$

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\begin{align*}
\{C, F\}, \{D, F\}, \{D, E\}, \{E, G\}
\end{align*}

If we choose to depict the edges in \( E \) as arcs then we may diagram \( G \) as follows:

The degree sequence of this graph is \( 4, 3, 3, 3, 3, 1 \) or \( 4, 3^5, 1 \).

**Definition 6.7** A depiction of a graph \( G(V, E) \) made by assigning members of \( V \) to distinct points in the plane and presenting members of \( E \) as arcs in the plane is called a **drawing** of a graph.

It is important to note that graphs do not care how they are drawn. One graph may have many drawings and these drawings can look quite different as shown in Example 6.2.

**Example 6.2** The graph \( G(V, E) \) with

\[
V = \{A, B, C, D\}
\]

and

\[
E = \{\{A, B\}\{A, C\}\{A, D\}\{B, C\}\{B, D\}\{C, D\}\}
\]

can be drawn in either of the following two (as well as many more) fashions:

**Definition 6.8** If all vertices of a graph have the same degree the graph is **regular**. If the common degree is \( d \) then the graph is said to be **\( d \)-regular**. Graphs that are \( 3 \)-regular are said to be **cubic graphs**.

The graph is Example 6.1 is not regular while the graph in Example 6.2 is \( 3 \)-regular.

**Definition 6.9** A walk of length \( n \) in a graph \( G(V, E) \) is an alternating sequence:

\[
v_1e_1v_2e_2v_3\cdots v_nv_nv_{n+1}
\]

of vertices and edges that begins and ends with a vertex and has the added property that \( e_i \) is incident with both \( v_i \) and \( v_{i+1} \).

**Definition 6.10** A path is a walk which does not repeat edges or vertices.

**Example 6.3** The graph below is given as a drawing so that the edges are deduced from the arcs joining the vertices \( A, B, C, D, E, \) and \( F \).

The sequence

\[
A - B - E - B - C - A - D - A - C
\]

is a walk that is not a path. The sequence

\[
A - B - C - F
\]

is an example of a path.

**Definition 6.11** A graph \( G(V, E) \) is said to be **connected** if for any two vertices \( u, v \in V \) there is a path from \( u \) to \( v \).

**Definition 6.12** Suppose the \( G(V, E) \) and \( H(U, F) \) are graphs. If \( U \subseteq V \) and \( F \subseteq E \) then we say \( H \) is a **subgraph** of \( G \) and write \( H \leq V \).

**Definition 6.13** A subgraph \( H \leq G \) is said to be **proper** if \( H \neq G \).

Notice that the requirement that \( H \) be a graph in Definition 6.12 forces the ends of every member of \( F \) to be in \( U \); while \( U \) may be any subset of \( V \), \( F \) may not be an arbitrary subset of \( E \) - it is restricted to those edges with both ends in \( U \). Once we pick some subset \( U \subseteq V \) there is a “largest” subgraph possible on the vertices in \( U \).
6.1. BASIC DEFINITIONS

**Definition 6.14** Suppose that $G(V,E)$ is a graph and that $U \subset V$. Then the graph $H(U,F)$ in which $F$ contains every member of $E$ with both ends in $U$ is called the **induced subgraph of $G$ on $U$**. If a subgraph of $G$ is the induced subgraph of $G$ on its own vertex set it is called an **induced subgraph**.

If we have a graph then either it is connected or it is made of a collection of maximal connected induced subgraphs. In this case “maximal” means in the partial order of subsets of the vertex set.

**Definition 6.15** The **connected components** of a graph are its maximal connected induced subgraphs. A connected graph has a single connected component comprising the entire graph.

**Definition 6.16** Suppose that $G(V,E)$ is a graph so that $V$ is the union of two disjoint sets $A$ and $B$. If every member of $E$ has one end in $A$ and the other end in $B$ then $G$ is a **bipartite graph** and $A$ and $B$ are said to form a **bipartition** of $V$.

**Example 6.4** In this example we give a graph by its drawing alone. If we are not overly concerned with the identity of individual vertices, this is often a convenient way to display a graph.

This graph is cubic with six vertices, nine edges, and it is bipartite. To see that the graph is bipartite, notice that taking every other vertex around the displayed hexagonal cycle in the graph yields a bipartition.

Another notation often used to specify the vertices and edges of a graph $G$ is to say $V(G)$ for the vertices and $E(G)$ for the edges. We use this notation in the following definition.

**Definition 6.17** Let $G$ be a simple graph. Suppose that $E$ is the set of all unordered pairs from $V$. Then the **compliment of $G$** is the graph $G'$ with $V(G') = V(G)$ (the same vertex set) and $E(G') = E - E(G)$ (the complimentary edge set). We denote the compliment of $G$ by $\overline{G}$.

**Definition 6.18** If $G(V,E)$ is a graph and $v \in V$ is a vertex then the set of vertices of $G$ that are adjacent to $v$ is called the **neighborhood of $v$**. The neighborhood of a vertex is denoted $\Gamma(v)$.

**Example 6.5** The graph below, with labeled vertices, is another drawing of the graph from Example 6.4. In this graph

$\Gamma(v) = \{x,y,z\}$

while the vertices $a$ and $b$ are not in the neighborhood of $v$. Notice that $v$ is not a member of its own neighborhood (since $v$ is not adjacent to itself).

![Graph Diagram]

**Problems**

**Problem 6.1** Prove Lemma 6.1 that says that the number of vertices in a finite simple graph of odd degree must be even. Deduce that the sum of the degrees of a finite simple graph must be even.

**Problem 6.2** In a graph the set of all ordered pairs of adjacent vertices form a set of ordered pairs and hence a relation. What properties (reflexive, symmetric, transitive, antisymmetric) must the relation have to be able to specify the unordered pairs forming the edges of a simple graph?

**Problem 6.3** Let $G$ be a finite simple graph with at least two vertices. Either prove that $G$ must have two vertices of equal degree or find a graph which is a counterexample.

**Problem 6.4** For each of the following lists of numbers, either produce a graph that is a witness that the
list is the degree sequence of a simple graph or prove that the list cannot be the degree sequence of a graph.

(i) 3,3,3,3,3,3
(ii) 5,5,5,5,5,5,5
(iii) 1,2,3,4,5,6
(iv) 1,2,3,4,4,5,7
(v) 3,3,3,5,5,5
(vi) 4,4,4,4,4,4

Problem 6.5 Give an exact formula for the minimal number of vertices in an n-regular simple graph. You must demonstrate that graphs attaining the minimal number of vertices exist.

Problem 6.6 Suppose that G is a finite simple graph. Prove that
\[ \frac{1}{2} \sum_{v \in V(G)} \delta(v) = |E(G)| \]

Problem 6.7 Suppose that G(V, E) is a graph with V\{2, \ldots, 16\} and E = \{\{a, b\} : a \neq b\}. Compute the number of connected components of G and their sizes.

Problem 6.8 Prove that the relation H is a subgraph of G is a partial order of the set of finite simple graphs.

Problem 6.9 Starting with the graph in Problem 6.7 give a drawing of the induced subgraph on the even vertices.

Problem 6.10 Let G(V, E) be a graph with V equal to the set of all strings over the alphabet \{0, 1\} of length 4. Define E by making two vertices adjacent if they differ in a single position. Prove or disprove:
- G is connected.
- G is bipartite.

Problem 6.11 Let G(V, E) be a graph whose vertices are the two-dimensional Cartesian plane and whose edges are pairs of points at Euclidean distance one. Prove or disprove:
- This graph is connected.

(ii) This graph is bipartite.

Problem 6.12 Let G(V, E) be a graph whose vertices are polynomials of degree at most three over the integers (mod 2). Define edges to be pairs of polynomials that share a root. Sketch a drawing of this graph and compute its number of connected components.

Problem 6.13 Examining a multiplication table of the non-zero integers (mod 13) demonstrates that only some numbers (mod 13) are perfect squares. Suppose that we make a graph G(V, E) given by:

\[ V = \mathbb{Z}_{13} - \{[0]_{13}\} \]
\[ E = \{\{a, b\} : a - b \text{ is a perfect square}\} \]
- Is this graph connected?
- Is this graph regular?
- Is this graph bipartite?

Problem 6.14 Let G(V, E) have

\[ V = \{ax + b : a \neq 0 \text{ with } a, b \in \mathbb{Z}_5\} \]

with two lines adjacent in G if the lines intersect at some point.

- How many vertices are there in this graph?
- How many edges are there in this graph?
- Is this graph connected?
- Is this graph regular?
- Is this graph bipartite?

Problem 6.15 Let H be the compliment of the graph specified in Problem 6.14. Find the number of connected components and make a drawing of the graph. Does the drawing of the compliment make it easier to understand the original graph?

Problem 6.16 If G is a finite simple graph prove that if G is not connected then \(\overline{G}\) is connected.

Problem 6.17 Give an example of a simple graph G with more than one vertex so that both G and \(\overline{G}\) are connected and G has as few vertices as possible.
6.2. EXAMPLES OF GRAPHS

Problem 6.18 Let $G$ be a finite simple connected graph in which there is a unique path between any two vertices. Prove directly (without added terminology or definitions from outside this section) that the graph is bipartite.

Problem 6.19 Suppose that we have two simple graphs $G$ and $H$ so that $H$ is the complement of $G$. What is the smallest number of vertices that force either $G$ or $H$ to have three mutually adjacent vertices?

Problem 6.20 Let $Q$ be a regular octahedron. Define $G$ so that $V(G)$ are the faces of the octahedron and $E(G)$ are pairs of faces that share an edge.

- Is this graph connected?
- Is this graph regular?
- Is this graph bipartite?

Problem 6.21 Suppose we have an infinite graph $G(V, E)$ with $V = \mathbb{N}$ and edges being pairs of numbers that differ by multiples of 2. What set is $\Gamma(0)$?

Problem 6.22 If $G(V, E)$ is the graph given in Problem 6.11 then what is $\Gamma(0, 0)$?

Problem 6.23 Find a graph $G$ so that the induced subgraph on the neighborhood of every vertex in $G$ is the same as the graph shown above. Use as few vertices as possible.

6.2 Examples of Graphs

This section introduces a large number of known graphs that form a library of examples, counterexamples, and useful parts for solving graph theoretic problems. All graphs defined in this section are simple graphs.

Definition 6.19 A graph on $n$ vertices with all possible edges is called the complete graph on $n$ vertices and is denoted $K_n$. Example 6.6 shows three examples of complete graphs.

Example 6.6 Shown below are drawings of the complete graphs on 5, 6, and 7 vertices.

![Complete Graphs](image)

Definition 6.20 A bipartite graph in which one part of the partition contains $n$ vertices, the other contains $m$ vertices, and which has all possible edges is called the complete bipartite graph on $n$ and $m$ vertices and is denoted $K_{n,m}$. The graph in Example 6.4 is a drawing of $K_{3,3}$.

Definition 6.21 An n-cycle is a graph on $n \geq 3$ vertices where the vertices are arranged in a ring and each is adjacent to its two neighbors. The notation for a cycle is $C_n$ and $n$ is said to be the length or size of the cycle. Examples of cycles are given in Example 6.7.

Example 6.7 Shown below are drawings of cycles on 7, 12, and 20 vertices.

![Cycles](image)

Definition 6.22 A path of length $n$ is a simple graph on $n \geq 3$ vertices that consists of a single path with $n$ vertices. A path of length $n$ may be obtained by deleting any one edge from an $n$-cycle. A path of length $n$ is denoted $P_n$.

Definition 6.23 Let $\Delta \subset \mathbb{Z}_n$. Then an $n$-difference graph with differences $\Delta$ is a graph $G$ with $V(G) = \mathbb{Z}_n$ and

$$E(G) = \{\{a, b\} : a - b \in \Delta\}$$

Notice that if we take $\Delta = \{1\}$ then an $n$-difference graph is identical to a cycle while if we take $\Delta$ to be every nonzero number $(\text{mod } n)$ then an $n$-difference
graph is identical to a complete graph. The world identical is currently being used in an intuitive fashion. We will make it precise in the next section.

**Definition 6.24** A cycle in a graph G is a subgraph of G that is, itself, a cycle. A cycle with n vertices is called an n-cycle. A graph with no cycles in it is said to be acyclic.

**Definition 6.25** A tree is a connected acyclic graph. An example of a tree appears in Example 6.8.

**Example 6.8** Notice that the graph pictured below is connected and has no cycles. It thus satisfies the definition of a tree.

![Diagram of a tree](image)

**Lemma 6.2** A finite tree must contain a vertex of degree less than two.

**Proof:**

Suppose that T is a tree such that the minimum degree of any vertex in T is two. First, notice that this forces the tree to have at least 3 vertices (K₃ is 2-regular and has as many edges as possible). Starting at any vertex, build a walk by repeating the following steps: pick an adjacent vertex you have not yet visited and add the relevant edge and corresponding vertex to the walk. Continue until the current vertex is adjacent to a vertex in the walk other than the immediately preceding vertex. To see that this is possible, notice that since the minimum degree of all vertices is two, during a walk any vertex you enter will be able to leave at least once. This permits you to continue until all adjacent vertices are ones you have visited before, (notice that the graph is finite). Finally notice that this procedure locates a cycle. Since T is, by definition, acyclic the initial hypothesis that the minimum degree is two is in error and the lemma follows. □

**Corollary 6.1** Any finite tree with two or more vertices contains a vertex of degree one.

**Proof:**

Notice that a connected graph on more than one vertex has at least one edge incident with each vertex. This is required for the paths from that vertex to the others that are witnesses that the graph is connected. Lemma 6.2 thus implies that the minimum degree of a tree with more than one vertex is exactly one. □

**Lemma 6.3** A tree with n vertices has n − 1 edges.

**Proof:**

We proceed by induction. Notice that, since no edges are possible, there is a unique graph on one vertex. This graph has no cycles (a cycle requires at least 3 vertices) and so is acyclic. It has a single connected component consisting of a single vertex. The one vertex graph is thus connected and acyclic which means that it is a tree. Notice is has 1 − 1 = 0 edges and so the lemma is true for n = 1. Assume that the lemma is true for a tree with n vertices, for some value of n.

If T is a tree on n + 1 vertices, Corollary 6.1 tells us that some v ∈ V(T) has degree one. If we create a new graph T’ by deleting v and its single incident edge from T, the resulting graph is still connected (none of the paths witnessing the connections between other vertices could have run through v). Deleting a vertex and an associated edge could not create a cycle in T” so we see T’ is connected and acyclic. This means that it is a tree with n vertices and hence n − 1 edges by the induction hypothesis. Given how we constructed T’, T has one more edge and vertex than T’ meaning T, is a tree on n + 1 vertices and has n edges. Since T was an arbitrary tree on n + 1 vertices, the Lemma follows by induction. □

**Definition 6.26** The leaves of a tree are the vertices of degree one.

**Definition 6.27** A forest is a graph in which each connected component is a tree.

With the notion of a cycle in a graph defined it becomes possible to prove the following handy fact about bipartite graphs.

**Theorem 6.1** All the cycles in a bipartite graph are of even length.
6.2. EXAMPLES OF GRAPHS

Proof:

Let $G$ be a bipartite graph with bipartition $V(G) = A \cup B$. Let $C$ be a cycle in $G$. Since $C$ contains edges it contains at least one vertex $v \in A$. Because all edges of $G$ have one end in $A$ and the other in $B$, the cycle must alternate between these sets. If we start at $v$ and go around $C$ the vertices thus have the pattern $A, B, A, B, \ldots$. Since the last vertex before $v$ as we go around the cycle must be in $B$ the vertices can be grouped into disjoint pairs with the first from $A$ and the second from $B$; this forces an even number of vertices in $C$, making $C$ a cycle of even length. $\Box$

The converse of this theorem is also true but we will delay giving the proof until we have coloring theory to help us.

We now move on to constructions that creates new graphs from old ones.

**Definition 6.28** The prism of a graph $G$ is obtained by making two copies of $G$ and adding edges that have the corresponding vertices in each copy as their ends. The prism of the 5-cycle is shown in Example 6.9.

**Example 6.9** Shown below are drawings of the 5-cycle and its prism.

![5-cycle and prism](image)

There is a family of graphs called prisms which are defined as follows.

**Definition 6.29** The $n$-prism is the prism of the $n$-cycle. The second graph shown in Example 6.9 is the 5-prism.

Once we have the notion of the prism of a graph then it is possible to iterate the prism operation to create a famous family of graphs.

**Definition 6.30** The $n$-hypercube or $n$-cube is defined recursively. The 0-hypercube consists of a single vertex. The $n+1$-hypercube is the prism of the $n$-hypercube. Drawings of hypercubes appear in Example 6.10. The $n$-cube is denoted as $H_n$.

**Example 6.10** Shown are drawings of the 1-cube, 2-cube, 3-cube, and 4-cube. Notice that there are multiple ways to derive the next cube as a prism. The 3-cube, for example, has three opposite pairs of faces.

![Hypercubes](image)

**Definition 6.31** The girth of a graph is the length of the shortest cycle that appears as a subgraph.

The graph in Example 6.11 is so famous that there is a book named after it. It is famous because it is a counter-example to a large number of conjectures in graph theory as well as a minimal example for a number of properties. It is, for example, the smallest 3-regular (cubic) graph of girth 5.

**Example 6.11** The graph drawn below is called the Petersen graph. It is a 3-regular graph of girth five with ten vertices.
The following construction is a little hard to follow at first and so comparing the definition with Example 6.12 may help.

**Definition 6.32** The **generalized Petersen graph** $P_{n,m}$ is a graph with $2n$ vertices. The edges may be specified with arithmetic (mod $n$). Make two copies of $\mathbb{Z}_n$ with elements of the first copy denoted $0, 1, 2, \ldots$ and elements of the second copy being denoted $0', 1', 2', \ldots$. The edges come in three groups: $\{x, x+1\}$, $\{x', x'+m\}$, and $\{x, x'\}$. This latter group of edges are called the **spokes**. The Petersen graph is the generalized Petersen graph $P_{5,2}$.

**Definition 6.33** The **line graph** of a graph $G(V,E)$ is a new graph $L$ with

$$V(L) = E(G)$$

and

$$E(L) = \{\{e, f\} : |e \cap f| = 1\}$$

In other words, the vertices of $L$ are the edges of $G$ and two edges of $G$ are adjacent as vertices of $L$ if they share one vertex. We denote the line graph of $G$ by $L(G)$.

**Example 6.12** Compute the line graph of the complete bipartite graph $K_{3,3}$. We start by making a labeled drawing of $K_{3,3}$ that will let us name all the edges.

The edges of this graph are: $AB, AC, AD, BD, BE, CD, CE, DF,$ and $EF$. The three edges incident at each vertex form a triangle in the new graph. This suggests the following drawing with two sorts of triangles neatly meshed.

The triangles associated with the edges incident on $A$, $D$, and $E$ are displayed as explicit triangles while those associated with $B$, $C$, and $F$ are displayed radially with a single curving edge.

The line graph of $K_{3,3}$ is an example of another construction.

**Definition 6.34** Suppose that $G(V,E)$ and $H(U,F)$ are both graphs. Then the **Cartesian product** of $G$ and $H$, denoted $G \times H$ has vertex set $V \times U$ and edges given as follows. For

$$(a, b), (c, d) \in V(G \times H),$$

we have that $\{(a,b),(c,d)\}$ is an edge of $G \times H$ if either $b = d$ and $(a,c) \in E(G)$ or $a = c$ and $(b,d) \in E(H)$. The line graph of $K_{3,3}$ in Example 6.12 is an example of $C_3 \times C_3$, the Cartesian product of two copies of the 3-cycle.
6.2. EXAMPLES OF GRAPHS

Problems

Problem 6.24 Compute the number of edges in $K_n$.

Problem 6.25 Compute the number of edges in $K_{n,m}$.

Problem 6.26 Find all cycles that are (i) complete graphs, (ii) complete bipartite graphs, (iii) bipartite.

Problem 6.27 Find an exact formula for the maximum number of connected components in a 2-regular graph with $n$ vertices.

Problem 6.28 Prove that an $n$-difference graph is regular.

Problem 6.29 Find a nonempty set $\Delta$ of differences so that a 12-difference graph is a simple graph with more than one connected component. Explain why the requirement that the graph be simple prevents zero from being in $\Delta$.

Problem 6.30 Prove or disprove. If $p$ is prime and $\Delta$ is a nonempty set of differences not containing zero then a $p$-difference graph with differences $\Delta$ is connected.

Problem 6.31 Suppose that $G$ is an $n$-difference graph with differences $\Delta$ and that the set $\Delta$ is not closed under negation (mod $n$). If $\Delta'$ is the closure of $\Delta$ under negation and $H$ is an $n$-difference graph with differences $\Delta'$ then what is the relationship between $G$ and $H$?

Problem 6.32 Prove that all induced cycles of $K_n$ are 2-cycles. An induced cycle is an induced subgraph that is a cycle.

Problem 6.33 Find all regular trees and provide drawings of them.

Problem 6.34 Suppose that $T$ is a tree with $n$ vertices. Prove that

$$\sum_{v \in V(T)} \delta(v) = 2n - 2$$

Problem 6.35 Suppose that $F$ is a forest with $n$ vertices and $k$ connected components. Give an exact formula for the number of edges in $F$.

Problem 6.36 For which $n$ is the $n$-cube

(i) a complete graph,

(ii) a cycle,

(iii) a generalized Petersen graph?

Problem 6.37 Compute the number of 4-cycles that appear as subgraphs of $H_n$.

Problem 6.38 Compute the number of 6-cycles that appear as subgraphs of $H_n$.

Problem 6.39 Compute the number of 3-cubes that appear as subgraphs of $H_n$ for $n \geq 3$.

Problem 6.40 Find an induced subgraph of $H_5$ that has the largest possible number of vertices and is a path.

Problem 6.41 Draw the generalized Petersen graph $P_{8,3}$ and compute its girth.

Problem 6.42 Draw the generalized Petersen graph $P_{8,2}$ and compute its girth.

Problem 6.43 Prove that the girth of a generalized Petersen graph is at most eight.

Problem 6.44 Find a generalized Petersen graph of girth eight (and prove your answer is correct).

Problem 6.45 If $G$ is a $k$-regular graph on $n$ vertices, compute the number of vertices and edges in its line graph $L(G)$.

Problem 6.46 Compute the line graph of the cycle $C_n$.

Problem 6.47 Compute the line graph of a path on $n$ vertices.

Problem 6.48 Give necessary and sufficient conditions on a graph $G$ for $L(G)$ to contain a 3-cycle.

Problem 6.49 Notice that we can iterate the line graph construction, generating a sequence of graphs $G, L(G), L(L(G)), \ldots$ Give an example of a tree for which:

(i) The sequence of sizes of graphs goes to zero.
(ii) The sequence of sizes of graphs goes to a fixed positive value.

(iii) The sequence of sizes of graphs diverges to infinity.

**Problem 6.50** Find a drawing of $\mathcal{L}(K_4)$ in which no two of the arcs representing edges intersect, except at the vertices. This is called a non-crossing drawing.

**Problem 6.51** Compute, as a function of $n$ and $m$ the girth of $C_n \times C_m$. Recall that a cycle must have at least three vertices.

**Problem 6.52** Prove or disprove that $G \times K_2$ is the same graph as the prism of $G$.

**Problem 6.53** Prove or disprove that $C_4 \times C_4$ is the same graph as $H_4$.

**Problem 6.54** Suppose that $G$ and $H$ are graphs. Give formulae for the number of vertices and edges of $G \times H$ in terms of the number of vertices and edges in $G$ and $H$.

**Problem 6.55** Prove that the Cartesian product of regular graphs is regular.

**Problem 6.56** Suppose the Cartesian product of two graphs $G$ and $H$ is connected. Can we deduce anything about the connectedness of $G$ and $H$?

### 6.3 Isomorphism and Automorphism

In the last section we promised to define precisely what it means for two graphs to be the same. In earlier chapters we developed a notion of what it meant for two groups or rings to be isomorphic. In both cases an isomorphism was a bijection that preserved the group or ring operations. Another way to view this is that isomorphisms are bijections that preserve structure.

**Definition 6.35** An **isomorphism** of two graphs $G(V,E)$ and $H(U,F)$ is a bijection of $V$ and $U$ that takes edges of $G$ to edges of $H$ and vice versa (the inverse of the bijection takes edges of $H$ to edges of $G$). Formally $\Phi : V \to U$ is a graph isomorphism if $\Phi$ is a bijection that satisfies the property: $\{a,b\} \in E$ if and only if $\{\Phi(a),\Phi(b)\} \in F$. If an isomorphism exists between two graphs we say they are isomorphic. We denote isomorphism of graphs by $G \cong H$.

Two graphs are the same as graphs if they are isomorphic. We will illustrate this in the homework problems by producing several alternate representations of graphs defined in Section 6.2. There are several obvious properties of isomorphic graphs.

**Lemma 6.4** Suppose that $G \cong H$ are isomorphic finite graphs.

(i) $|V(G)| = |V(H)|$,

(ii) $|E(G)| = |E(H)|$,

(iii) $G$ and $H$ have the same degree sequence.

**Proof:**

Property (1) follows from the fact that bijections of finite sets are size preserving. For property (2) notice that the isomorphism must take edges to edges in both directions. This means that the map from edges to edges is also a bijection and so preserves size (the number of edges). For property 3, let $\Phi$ be a graph isomorphism and note that a vertex $v$ and its neighbors $\Gamma(v)$ must, in order to preserve edges, be mapped to a vertex $\Phi(v)$ and its neighbors $\Gamma(\Phi(v))$. This means the isomorphism takes vertices of a given degree to another vertex of the same degree and so must preserve the degree sequence.

Computing if two graphs are isomorphic is a difficult problem, though its exact difficulty is unknown to computational complexity theorists. Computing that two graphs are not isomorphic is often quite easy and almost all possible “non-isomorphism” proofs can be performed with the tools in Lemma 6.4. Two graphs with different numbers of edges or vertices cannot be isomorphic. Two graphs with different degree sequences cannot be isomorphic. Likewise, when searching for an isomorphism, vertices must be mapped to vertices of the same degree. The most difficult case for determining isomorphism is between graphs that have the same degree sequence.

**Definition 6.36** An **invariant** of a graph is a quality of the graph that is preserved by isomorphism.
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Size of vertex set, number of edges, and degree sequence are all examples of graph invariants.

Example 6.13 The top two graphs below are isomorphic (they are both drawings of \( C_7 \)) while the lower two are not isomorphic. None of the tests implicit in Lemma 6.4 separate these two graphs: both have eight vertices and edges and degree sequence \( 2^8 \). One of the graphs is \( C_8 \) while the other is two copies of \( C_4 \). The lemma following this example captures an invariant that distinguishes the lower pair of graphs.

![Graphs](image)

**Lemma 6.5** The number of connected components in a graph is a graph invariant.

Proof:

Suppose that \( G \) is a graph with \( n \) connected components and \( H \) is a graph with \( m \) connected components with \( n \neq m \). Without loss of generality we may assume that \( n > m \). Suppose that \( f : V(G) \to V(H) \) is an isomorphism of \( G \) with \( H \). Let \( R = \{v_1, v_2, \ldots, v_n\} \) be a subset of \( V(G) \) containing one vertex from each connected component. Then, by the pigeonhole principle two members of \( R \), \( v_i \) and \( v_j \) will be mapped to the same connected component in \( H \). Let \( P \) be a a path from \( f(v_i) \) to \( f(v_j) \) in \( H \). Since the definition of isomorphism requires that \( f \) and \( f^{-1} \) map edges to edges, \( P \) has a preimage in \( G \) and so \( v_i \) and \( v_j \) are in the same connected component of \( G \). We have achieved a contradiction and so the supposition that the isomorphism \( f \) existed was incorrect and the lemma follows. \( \square \)

The next lemma is broadly useful for establishing that two graphs are not isomorphic and may be useful for guiding the search for potential isomorphisms when they are isomorphic.

**Lemma 6.6** The presence of a given subgraph is a graph invariant.

Proof:

Suppose that \( f : G \to H \) is an isomorphism of graphs \( G \) and \( H \) and that \( S \) is a subgraph of \( G \). Then \( f(S) \) is a subset of \( V(H) \). Since \( f \) preserves edges it follows that each edge in \( S \) has an image between two corresponding vertices in \( f(S) \). It follows that there is a subgraph of \( H \) isomorphic to \( S \). \( \square \)

**Corollary 6.2** Girth is a graph invariant.

Proof:

Notice that a cycle-subgraph that is a witness to girth is a subgraph and so is preserved by isomorphism by Lemma 6.6. \( \square \)

One of the most powerful tools in mathematics is to shift ones point of view so as to make a previously difficult task (for example a proof or computation) simple. A representation of a graph is a method of presenting a graph, of specifying its vertices and edges. The next lemma gives an alternate presentation of the hypercube that will prove useful later in the chapter.

**Theorem 6.2** Let \( S = \{1, 2, \ldots, n\} \). Define a graph \( G_n(V, E) \) by

\[
V(G_n) = \mathcal{P}(S) \quad \text{(the power set of } S) \]

and

\[
E(G_n) = \{\{A, B\} : |A \Delta B| = 1\}
\]

Recall that \( \Delta \) is the symmetric difference operator on sets. Then \( G_n \cong H_n \).

Proof:

We proceed by induction on \( n \). If the set \( S \) is empty then \( G \) is an isolated vertex (the power set of the empty set is the empty set). We note that \( H_0 \)
is also a graph consisting of a single vertex. Thus $G_0 \cong H_0$. If the set $S = \{1\}$ then $G$ contains two vertices one corresponding to the empty set and one corresponding to the set $\{1\}$, the symmetric difference between these two sets is the set $\{1\}$ and hence has size 1, thus the resulting graph is a single edge connecting two vertices and it follows that $G_1 \cong H_1$. Suppose that the theorem is true for some $n \geq 1$ so that $G_n \cong H_n$.

Partition $\mathcal{P}([1, 2, \ldots, n, n + 1])$ into two non-empty disjoint sets: $L$ consisting of sets that do not contain $n + 1$ and $R$ consisting of sets that do contain $n + 1$. The induced subgraph of $G_{n+1}$ on $L$ is exactly $G_n$. The induced subgraph of $G_{n+1}$ on $R$ is isomorphic to $G_n$. To see this notice that deleting $n + 1$ from the elements of $R$ yields a bijection of $R$ with $L$ that exactly preserves the edge relation: that sets differ by one element. Note that the pairs mapped to one another by this bijection of $L$ and $R$ themselves differ by the one element $n + 1$ and so are the ends of edges of $G_{n+1}$. Note also that no other edges of $G_{n+1}$ have an end in $L$ and an end in $R$ because all such pairs already differ by $\{n + 1\}$. This means that the edges of $G_{n+1}$ that go between $L$ and $R$ complete a prism using the copies of $G_n$ on $L$ and $R$. Recall that $H_{n+1}$ is the prism of $H_n$. Since $G_n \cong H_n$ by the induction hypothesis it follows that $G_{n+1} \cong H_{n+1}$ and the Theorem follows by induction. □

**Example 6.14** The digraph below shows pictorially the isomorphism derived in Theorem 6.2 for $n = 3$.

The representation of hypercubes as a system of sets gives us an alternate point of view that may make it easier to prove some properties of hypercubes. An example of this appears in the following lemma.

**Lemma 6.7** The hypercube $H_n$ is bipartite.

**Proof:**

Using the representation of the hypercube given in Theorem 6.2, we notice that the ends of any edge are sets differing by a single element. This means one end is a set of even size while the other is a set of odd size. If we partition the vertices into sets of even and odd size respectively then all edges have one end in each of these sets. This demonstrates that the graph is bipartite with the bipartition formed of sets of even and odd size. □

We now turn to an application of groups from Chapter 4.

**Definition 6.37** An isomorphism of a graph with itself is called an automorphism.

**Theorem 6.3** The set of all automorphisms of a graph form a group under the operation functional composition.

**Proof:**

A automorphisms of a graph $G$ must be a bijection of the vertex set with itself. This means that the set of all automorphisms of a graph is a subset of the group $\text{Sym}(V(G))$ of all bijections of the vertex set with itself. Suppose we have two automorphisms $\sigma$ and $\tau$. Then both maps take edges to edges; this means that their functional composition also takes edges of $G$ to edges of $G$. This means the set of automorphisms of $G$ is closed under functional composition. The definition of isomorphism requires that the inverse of an isomorphism also preserve edges and so the set of automorphisms is closed under functional inversion. Since this is the notion of inverse used in $\text{Sym}(V(G))$ we see the set of automorphisms is also closed under inversions. This is sufficient to make the set of all automorphisms a subgroup of $\text{Sym}(V(G))$ and hence a group in its own right. □.

The group of all automorphisms of a graph $G$ is denoted $\text{Aut}(G)$. Note that the identity of $\text{Aut}(G)$ is the identity map, the map that takes each vertex to itself. The automorphism group of a graph is also called the symmetry group or symmetries of the graph.
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**Definition 6.38** A graph $G$ is said to be **rigid** if $|\text{Aut}(G)| = 1$, in other words the only automorphism is the identity map.

**Return of the Orbit-Stabilizer Lemma**

At this point we will develop a technique for computing the size of $\text{Aut}(G)$ that relies on the material from Chapter 4 on the action of groups on sets, Section 4.11. The automorphism group of a graph acts on the vertex set. This means that the orbit-stabilizer lemma (Corollary 4.13 can be used to compute the size of the group).

**Example 6.15** Prove that $\text{Aut}(P_n) \cong (\mathbb{Z}_2, +)$.

*Solution:*

Lemma 6.4 tells us that an automorphism (which is a type of isomorphism) preserves degree sequences. In $P_n$ there are exactly two vertices of degree one, the ends of the path. Name the ends of the path $u$ and $v$. This means that an automorphism can either leave $u$ and $v$ where they are or it may exchange them. This means that $|\text{Orbit}(u)| \leq 2$. Notice that the map $\tau$ that flips the path end-for-end is a bijection of edges that preserves the edges of the path an so is an automorphism. Notice that $\tau$ also exchanges $u$ and $v$ telling us that $|\text{Orbit}(u)| = 2$. Look at the automorphisms that fix $u$. Since $u$ may not move, the vertex next to $u$ cannot move (if it did this would break an edge). This effect continues down the path with the result that fixing $u$ forces the entire path to hold still. That means the stabilizer of $u$, $\text{Aut}(P_n)_u$, consists of only the identity map. Applying the orbit-stabilizer lemma we see that:

$$|\text{Aut}(P_n)| = |\text{Orbit}(u)| \cdot |\text{Aut}(P_n)_u| = 2 \cdot 1 = 2$$

and so $\text{Aut}(P_n)$ is a group of size two. Since the only group of size 2 is $(\mathbb{Z}_2, +)$, we are done.

One thing that the example shows is that the only symmetry that a path has, other than being left alone, is being flipped end-for-end. This is not too much symmetry. In many cases we can reuse reasoning connected with investigations of groups to compute automorphism groups of graphs.

**Definition 6.39** A graph $G$ is said to be **vertex transitive** if the action of $\text{Aut}(G)$ on $V(G)$ is transitive. In other words if for each $u, v \in V(G)$ there is some automorphism $\sigma \in \text{Aut}(G)$ so that $u \sigma = v$ ($\sigma$ takes $u$ to $v$).

In a vertex transitive graph, the automorphism group has a single orbit. This means that the size of $\text{Aut}(G)$ will be the size of $V(G)$ times the size of the group of automorphisms that fix any one vertex. The next example illustrates this.

**Example 6.16** Show that the dihedral group $D_n = \text{Aut}(C_n)$.

*Solution:*

Notice that the (geometric) vertices and edges of a regular $n$-gon have the same incidence structure as the (graph theoretic) vertices and edges of the cycle $C_n$. This means that the symmetries of a regular $n$-gon are also symmetries of $C_n$. This shows that $\text{Aut}(C_n)$ at least contains a copy of $D_n$ (contains, in this case, means “has a subgroup that is isomorphic to”). We now use the orbit-stabilizer lemma to compute the size of $\text{Aut}(C_n)$. First notice that we may rotate $C_n$ (in its presentation as a regular $n$-gon) to take any vertex to any other vertex. This means that $C_n$ is vertex transitive and hence for any $v \in V(G)$, $\text{Orbit}(v) = V(G)$.

Fix some $v \in V(G)$ and consider the automorphisms $\text{Aut}(C_n)_v$ that fix $v$. As in Example 6.15 the length of the shortest path between any vertex $u$ and the fixed vertex $v$ must be preserved - otherwise a map could not preserve the edges involved. Unlike the path example, we can flip the the cycle over in a manner that fixes $v$, but this is clearly the only automorphism, other than the identity map, that fixes $v$. Applying the orbit stabilizer lemma we get:

$$|\text{Aut}(C_n)| = |\text{Orbit}(v)||\text{Aut}(C_n)_v| = |V(C_n)| \cdot 2 = 2n$$

This means that the copy of $D_n$ in $\text{Aut}(C_n)$ is the whole automorphism group and we are done.

**Lemma 6.8** $\text{Aut}(K_n) \cong S_n$ (The automorphism group of the complete graph on $n$ vertices is the symmetric group on $n$ points).
Proof:

As we saw in the proof of Theorem 6.3, the automorphism group of a graph $G$ is a subgroup of $\text{Sym}(V(G))$. Since $K_n$ has all possible edges, every member of $\text{Sym}(V(K_n))$ is a member of $\text{Aut}(K_n)$. This means that $\text{Aut}(K_n) = \text{Sym}(V(K_n)) \cong S_n$ and we are done. \qed

**Lemma 6.9** A vertex transitive graph is regular.

Proof:

We already know that an isomorphism takes edges to edges and so must take a vertex to a vertex with the same degree. Since there is an automorphism that takes every vertex to every other vertex in a vertex transitive graph all the vertices have the same degree; the graph is regular. \qed

Next we turn to a construction that will give us a large class of vertex transitive graphs.

**Definition 6.40** Let $Q$ be a group and let $R = \{q_1, q_2, \ldots, q_k\}$ be a generating set for $Q$. Create a graph $G$ such that $V(G) = Q$ and $E(G) = \{\{g, h\} : g = q_i h\}$. We call $G$ a Cayley graph on $Q$ with generator set $R$. The notation for a Cayley graph is $\text{Cay}(Q, R)$.

Now that we have defined Cayley graphs it remains to show that they are vertex transitive.

**Lemma 6.10** Cayley graphs are vertex transitive.

Proof:

Let $G = \text{Cay}(Q, R)$ be a Cayley graph with $R = \{q_1, q_2, \ldots, q_k\}$. Examine the map $f : V(G) \to V(G)$ that takes $x \mapsto xq_i$. Notice that the map $y \mapsto yy_i^{-1}$ is its inverse, as a function, and so both these maps are bijections of $Q$. Suppose that $\{a, b\} \in E(G)$.

Then $a = q_i b$ for some generator $q_i$. Applying $f$ to both ends of this edge we get $aq_i$ and $bq_i = q_i^{-1}aq_i$. But then these two vertices are the ends of the edge $\{aq_i, bq_i\}$, since $q_i(bq_i) = aq_i$. This means that $f$ maps edges of $G$ to edges of $G$ and so is an automorphism.

This demonstrates that the maps given by right multiplication by generators are all automorphisms.

Since $R$ generates $Q$, every element of $Q$ is the product of a sequence of members of $R$.

Suppose that for some $q \in Q$ that $q = q_{i_1} \cdot q_{i_2} \cdots q_{i_m}$. Since the composition of automorphisms is an automorphism, right multiplication by $q$ is an automorphism of $G$. Notice that $q = e \cdot q$ so there is an automorphism of $G$ that takes $e$ to $q$. Since $q$ is an arbitrary member of $Q$ we deduce that $\text{Orbit}(e) = Q$. This in turn implies that $G$ is a vertex transitive graph. \qed

**Corollary 6.3** If $G = \text{Cay}(Q, R)$ is a Cayley graph then $\text{Aut}(G)$ contains a subgroup isomorphic to $Q$.

Proof:

Assume the situation of Lemma 6.10. In the proof of Lemma 6.10 we saw that right multiplication by members of $R$ on $Q = V(G)$ acts as an automorphism on $G$. Since $Q = \langle R \rangle$ it follows that all elements of $Q$ act as automorphisms in this fashion. Let $\iota : Q \to \text{Aut}(G)$ be the map that takes elements of $Q$ to the action by right multiplication on elements of $Q$. Since the action is exactly the group multiplication we have that

$$(x \cdot y)_\iota = x\iota \cdot y\iota$$

and we see that $\iota$ is a group homomorphism. From the proof of Lemma 6.10 we see that the elements of $q_i \in Q_i$ take $e \in Q_i$ to $q \in Q_i$ meaning that no two elements of $Q_i$ are equal. We deduce that $\iota$ is an injection. This gives us $Q \cong Q_i \leq \text{Aut}(G)$. \qed

**Example 6.17** Show that $\text{Cay} (\mathbb{Z}_n, \{1\}) \cong C_n$.

**Solution:**

Let $G = \text{Cay} (\mathbb{Z}_n, \{1\})$. Notice that the edges of $G$ incident on $x \in \mathbb{Z}_n$ are $\{x, x + 1\}$ and $\{x - 1, x\}$. Label the vertices of $C_n$ in order around the cycle with the labels $0, 1, \ldots, n-1$. Then the bijection that takes $\text{Vert}(G)$ to the vertices of $C_n$ that they label takes the edge $\{x, x + 1\} (\bmod n)$ in $E(G)$ to the edge $\{x, x + 1\} (\bmod n)$ in $E(C_n)$ and hence the bijection is an isomorphism. \qed

There is a connection in a Cayley graph between the generating set and the regularity of the graph. Notice that since edges in a graph are unordered pairs that
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having either or both of $\sigma$ and $\sigma^{-1}$ in a generating set has the same effect.

Notice also that the identity of the group is not included in the generating set if we wish the graph to be simple. Inclusion of the identity would generate a loop at each vertex.

**Definition 6.41** A generating set for a group is reduced if, for each element in the generating set, the inverse of that element is not present unless the element is its own inverse (is of order two).

**Lemma 6.11** If the simple graph $G = \text{Cay}(Q, R)$ is a Cayley graph and $R$ is a reduced generating set then the degree of vertices of $G$ is twice the number of element in $R$ of order greater than two plus the number of elements in $R$ of order two.

**Proof:**

Think of elements of $R$ as generating ways in or out of a vertex. A given element of $Q$ gets one edge for each generator and one edge corresponding to the inverse of the generator, unless the generator is of order two and hence its own inverse. The lemma follows. \(\square\)

We now look at another symmetry property a graph can have.

**Definition 6.42** A graph is edge transitive if there is some automorphism of the graph that can take any edge to any other edge. Equivalently, if the representation of the automorphism group on the edge set is transitive.

**Example 6.18** Edge transitive graphs.

- The graph $C_n$ is edge transitive. The automorphisms that rotate the cycle can take any edge to any other edge.
- The graph $K_n$ is edge transitive. The automorphism group is the symmetric group on $n$ points and so takes any pair of points to any other pair of points. It thus takes any edge to any edge.

**Problems**

**Problem 6.57** Find a path that is isomorphic to its own complement.

**Problem 6.58** Find a cycle that is isomorphic to its own complement.

**Problem 6.59** Prove that no cubic graph can be isomorphic to its complement.

**Problem 6.60** Prove that $K_{n,n}$ can be represented as (is isomorphic to) a difference graph.

**Problem 6.61** Consider the three generalized Petersen graphs $P_{7,1}$, $P_{7,2}$, $P_{7,3}$. For each pair either find an invariant that demonstrates the pair is not isomorphic or find an explicit isomorphism. It will help if you label the vertices when specifying an isomorphism.

**Problem 6.62** Prove that isomorphism is an equivalence relation on the set of graphs.

**Problem 6.63** For the graphs in Figure 6.1 determine which pairs of graphs are isomorphic. The results of Problem 6.62 may be helpful.

**Problem 6.64** Let $G$ be a graph in which $V(G)$ is the set of all pairs of numbers from $\{1, 2, 3, 4, 5\}$ and edges are all pairs in $V(G)$ that intersect in the empty set. Prove that this graph is isomorphic to the Petersen graph.

**Problem 6.65** Let $Q_n$ be a graph whose vertices are all $n$-character strings over the alphabet $\{0, 1\}$. If $n = 3$ the vertices are 000, 001, 010, 011, 100, 101, 110, 111, for example. Let the edges of $Q_n$ be pairs of strings that differ in exactly one position. Prove that $Q_n \cong H_n$.

**Problem 6.66** Prove that being bipartite is a graph invariant.

**Problem 6.67** Prove that $\text{Aut}(G) = \text{Aut}(\overline{G})$.

**Problem 6.68** Find a rigid graph with the smallest possible number of vertices.

**Problem 6.69** Compute the size of $\text{Aut}(K_{n,m})$ for $n > m$.

**Problem 6.70** Compute the size of $\text{Aut}(K_{n,n})$.

**Problem 6.71** Compute the automorphism group of the Petersen graph, to the point of figuring out which well known group it is isomorphic to. Problem 6.64 may be helpful.
Problem 6.72 Prove that $H_n$ is vertex transitive.

Problem 6.73 We have, including the representation in Problem 6.65, three representations of the hypercube $H_n$. Using whichever of these representations is most convenient show that

$$|\text{Aut}(H_n)| = n! \cdot 2^n$$

by using the following steps.

(i) Do Problem 6.72 and deduce that the orbit of any vertex under the action of Aut($H_n$) has size $2^n$.

(ii) Picking a representation and vertex wisely, demonstrate that the automorphisms fixing that vertex form a group isomorphic to $S_n$. Hint: embed the hypercube in $\mathbb{R}^n$ and consider what the automorphism group does to $\mathbb{R}^n$ when it moves the cube.

- Apply the orbit-stabilizer lemma to obtain the size of Aut($H_n$).

Problem 6.74 Problem 6.73 tells us that $|\text{Aut}(H_3)| = 3! \cdot 2^3 = 48$. In Example 4.50 we found that the number of rigid automorphisms of a (3-dimensional) cube is 24. Discuss why the geometric and graph-theoretic cubes do not have the same number of isomorphisms. Which automorphisms of $H_3$ are not automorphisms of the geometric cube?

Problem 6.75 This is a pretty hard problem, mostly because you need to work with wreath products. Show that Aut($H_n$) $\cong \mathbb{Z}_2 \wr S_n$ by finding an explicit correspondence. Hint: consider automorphisms of order 2 that exchange opposite faces of $H_n$.

Problem 6.76 Find the most general conditions on $n$ and $m$ that you can that make the generalized Petersen graph $P_{n,m}$ vertex transitive.

Problem 6.77 Find a Cayley graph that is isomorphic to $K_n$.

Problem 6.78 Prove that difference graphs are Cayley graphs.
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Problem 6.79 Corollary 6.3 demonstrates that Aut(Cay(Q,R)) contains a copy of Q. Find an example where the subgroup of Aut(Cay(Q,R)) isomorphic to Q is a proper subgroup.

Problem 6.80 Show that if G is a Cayley graph then so is the prism of G. Hint: apply a direct product to the underlying group.

Problem 6.81 Prove that Hₙ is a Cayley graph.

Problem 6.82 Suppose that Q is a group, G = Cay(Q,R) is a Cayley graph, and that σ ∈ Aut(Q). Prove that Cay(Q,R) ≅ Cay(Q,σR). In other words, prove that if we apply an automorphism of the group to the members of the generating set and generate a Cayley graph with this modified generating set, we get the same (isomorphic) graph.

Problem 6.83 Given that there are exactly three groups of order ten, Z₁₀, Z₅ × Z₂, and D₅, prove that the Petersen graph is not a Cayley graph. Problem 6.82 may reduce the size of this problem. Hint: start by proving an Abelian group will not do.

Problem 6.84 Prove that Cayley graphs are connected.

Problem 6.85 Make a drawing, in which edges do not cross, of

\[ \text{Cay}(S₄, \{(1 \ 2 \ 3 \ 4), (1 \ 2)\}) \]

Problem 6.86 Suppose that we let G = Cay(Q,R) in the usual fashion except that we relax the hypothesis that R generates Q. If (R) = S ≤ Q then what changes? In particular how many connected components does G have and how big are they?

Problem 6.87 Prove that if G = Cay(Q,R) and H = Cay(S,T) are both Cayley graphs then so is G × H.

Problem 6.88 Prove that Kₙ,m is edge transitive.

Problem 6.89 Find an example of a graph that is edge transitive but not vertex transitive with as few vertices as possible. Generalize this example to an infinite family of graphs that are edge transitive but not vertex transitive.

Problem 6.90 Prove or disprove: the line graph of an edge transitive graph is vertex transitive.

Problem 6.91 Prove that Hₙ is edge transitive.

Problem 6.92 Prove that the Petersen graph is edge transitive.

Problem 6.93 There are twelve 5-cycles in the Petersen graph. Prove that the action of the automorphism group on this set of pentagons is transitive (that the automorphism group can take any 5-cycle to any other).

6.4 Euler and Hamilton Cycles

In this section we will examine two famous types of cycles. Euler cycles which visit all the edges of a graph (many of which are not technically cycles because they re-use vertices) and Hamilton cycles which are cycles that visit every vertex in a graph. Hamilton cycles appear in the theory of computation - a fast algorithm for deciding if a graph has a Hamilton cycle would be extremely valuable, but is not currently known.

Figure 6.2: The Seven Bridges of Koenigsberg Problem

The original problem that motivated the notion of an Euler cycle was the famous seven bridge of Koenigsberg problem. The city had seven bridges joining the city on both sides of a river and on an island in the manner shown in Figure 6.2. An after-dinner game played by the citizens was to try to find a way to cross all seven bridges, once each, and arrive back at their starting point. The problem can be abstracted into a
multigraph (a graph with multiple edges) in the manner shown. The great mathematician Euler managed to demonstrate that the problem was unsolvable.

**Definition 6.43** An Euler Cycle in a graph is a walk in the graph that begins and ends at the same vertex and which visits each edge in the graph exactly once.

An Euler cycle is often given as a sequence of vertices with adjacent vertices in the sequence corresponding to edges, with the same vertex as the beginning and end of the sequence.

**Example 6.19** The graph below is a drawing of $K_{4,4}$. An Euler cycle in this graph is

```
01234567036147250
```

You should directly verify that all edges are used in this cycle.

Euler cycles turn out to have a simple existence criterion.

**Theorem 6.4** A finite graph has an Euler cycle iff all edges of the graph are in a single connected component of the graph and all vertices have even degree.

**Proof:**

We first note that having all edges in the same connected component is obviously necessary. Isolated vertices do not participate in an Euler cycle and so are not relevant. We thus assume that we are working with a connected graph $G$ in which all vertices are of even degree. In this case an Euler cycle may be constructed. Pick a vertex of $G$ and begin a walk that does not reuse edges. Since all vertices are of even degree, any vertex that we are able to enter must also have an edge that permits us to leave, with the exception of the original vertex that possessed odd unused degree as the walk left it. This means we will first become stuck when we arrive back at the original vertex.

There are two possibilities at this point: the walk is an Euler cycle or unused edges remain. In the former case we are done, in the latter there exist unused edges. Since the initial walk has uniformly even degree, the remaining unused degree at every vertex is even. Starting at any vertex with remaining unused degree, we may construct another walk that also ends at that vertex in the same fashion that the initial walk was constructed. Splice this walk into the original walk at its beginning and ending point. The result both places more edges into the walk we are constructing and reduces the number of unused edges. This means that the number of edges left after each iteration of the process for splicing in a new cycle is a decreasing sequence of non-negative integers. We will thus arrive at zero in some finite number of steps and have an Euler cycle. We now know that having uniformly even degree implies the existence of an Euler cycle in a connected graph.

Suppose we have vertices of odd degree. Each time a walk passes though such a vertex the degree is reduced by two and so eventually reaches one at which point the walk must be trapped at the vertex with the final edge. This requires that an odd vertex be the beginning of the cycle and so also its end. Lemma 6.1 tells us that the number of vertices of odd degree is even and so the presence of one such vertex compels the existence of another. This forces the cycles to have two beginnings and endings, which is impossible. □.

Notice that the proof of Theorem 6.4 works equally well for graphs and multigraphs. It also demonstrates that the seven bridges problem is almost maximally hopeless: all the vertices are of odd degree.

Notice that Example 6.19 decomposes the edges of $K_{4,4}$ into an 8-cycle around the outside and a second proceeding by jumps of size three around the interior, both of which begin and end at vertex 0. There is an object similar to an Euler cycle that relaxes the condition that the walk have the same beginning and ending.
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Definition 6.44 An Euler Path is a walk in a graph which visits each edge in the graph exactly once. Note that an Euler path is not, technically, a path.

Corollary 6.4 A finite graph has an Euler path iff all edges of the graph are in a single connected component of the graph and all vertices have even degree or of exactly two have an odd degree. In the latter case, the vertices of odd degree must be the beginning and ending of the path.

Proof: This proof is an exercise.

Example 6.20 Can the shape on the left below be drawn without lifting your pencil?

\[ \text{Solution:} \]

Create a graph in which all changes of direction of the pencil or places where lines meet are vertices. Such a graph is shown on the right, above. The degree sequence of the graph is \(3^2\cdot 3\) and so there is an Euler path that begins and ends at the two vertices of degree three. The shape may thus be drawn by first executing the upper horizontal stroke and then preceding about the outside of the drawing.

We are now ready to shift our focus to Hamilton cycles.

Definition 6.45 A Hamilton cycle in a graph is a cycle that includes all the vertices of the graph.

Example 6.21 The picture below shows \(H_3\) with edges drawn using dotted lines, except that the edges of a Hamilton cycle are also drawn with solid lines.

Definition 6.46 A graph that has a Hamilton cycle is said to be Hamiltonian.

It is a famous result, that you are asked to verify in Problem 6.103, that the Petersen graph is not Hamiltonian. It is one of only four known vertex transitive graphs that are not Hamiltonian. There is a graph construction that will let us build another vertex transitive non-Hamiltonian graph (see Problem 6.104).

Definition 6.47 In a graph \(G\), Simplexifying a vertex \(v \in V(G)\) consist of replacing that vertex with \(\delta(v)\) vertices each of which is adjacent to one of the neighbors of \(v\) and all of which are adjacent to one another. Simplexifying a whole graph consists of simplexifying all the vertices simultaneously. When simplexifying all the vertices of a graph \(G\) simultaneously we require that for an edge \(\{a, b\} \in E(G)\) there is a corresponding edge \(\{a', b'\}\) in the new graph such that \(a'\) and \(b'\) are simplexified vertices of \(a\) and \(b\) respectively. Simplexifying a cubic graph is called triangulating it.

Example 6.22 The picture below shows \(K_{3,3}\) and its simplexification or triangulation.

Lemma 6.12 A cubic graph is Hamiltonian iff its triangulation is also Hamiltonian.

Proof: This proof is left as an exercise.

In addition to the Petersen graph and its triangulation there are two other known graphs (also a graph
and its triangulation) that are vertex transitive but not Hamiltonian. This graph is the Coxeter Graph shown in Example 6.23. The fact that the Coxeter graph is both vertex transitive and non Hamiltonian is offered without proof.

**Definition 6.48** The generalized Coxeter graph \( C_{n,u,v} \) is a graph on \( 4n \) vertices. It is similar to the Petersen graph in that there are cycles on \( n \) vertices that use different jump lengths. There are three such cycles with jump lengths \( 1, u, \) and \( v \). There are also \( n \) additional vertices. Each of these is assigned to a position within the cycles and is adjacent to the corresponding vertices from all three cycles. The Coxeter graph is \( C_{7,2,3} \) and is shown in Example 6.23.

**Example 6.23** The Coxeter graph:

![Graph Image]

We now turn to an application of Hamilton cycles called Gray codes. They are named after their inventor, Frank Gray of Bell Labs.

**Definition 6.49** A Gray code is a list of all the binary strings of length \( n \) so that adjacent words differ in exactly one position. The first and last element of the list are also required to disagree in exactly one position.

**Example 6.24** A Gray code for \( n = 3 \) is shown below.

| 000 | 001 | 011 | 010 | 110 | 111 | 101 | 100 |

**Lemma 6.13** A Gray code on \( n \)-character binary strings is equivalent to a Hamilton cycle in \( H_n \).

Proof:

Adopt the string representation for \( H_n \) derived in Problem 6.65. Then adjacent vertices are binary strings of length \( n \) that differ in one position. A Hamilton cycle viewed in this representation may be used to generate the list required starting at any vertex and vice versa.

We note that once we have a Gray code we may use it to list all subsets of a given set efficiently. If the all-zeros word corresponds to the empty set and the 1s in a word tell us which set elements are present in a subset then following a Gray code permits us to add and delete single elements from a working set and efficiently enumerate all subsets. A similar algorithm permits a Gray code to enumerate all subspaces of a vector space over \( \mathbb{Z}_2 \) by adding and subtracting basis elements. This is useful in algorithms connected with coding theory.

**Problems**

**Problem 6.94** Compute a minimal number of bridges to be added to the seven bridges problem to permit (i) an Euler path (ii) an Euler cycle. Give a drawing showing where your added bridge(s) would appear.

**Problem 6.95** List all complete graphs that have Euler cycles.

**Problem 6.96** Construct an Euler cycle in \( K_7 \) on the vertex set \( \{0, 1, 2, 3, 4, 5, 6\} \).
6.4. EULER AND HAMILTON CYCLES

**Problem 6.97** List all complete bipartite graphs that have Euler cycles.

**Problem 6.98** Prove Corollary 6.4. Be sure to draw on the proof of Theorem 6.4.

**Problem 6.99** Which of the shapes above can be drawn without lifting your pencil? Construct the graphs as in Example 6.20, give the degree sequence, and state the conclusion.

**Problem 6.100** Make a complete list of all graphs with induced Hamilton cycles.

**Problem 6.101** Prove that the prism $P_n$ is Hamiltonian.

**Problem 6.102** Using the iterated prism representation of the Hypercube, constructively prove that $H_n$ is Hamiltonian for all $n \geq 2$.

**Problem 6.103** Prove that the Petersen graph is not Hamiltonian.

**Problem 6.104** Prove Lemma 6.12.

**Problem 6.105** Prove that the triangulation of the Petersen graph is a vertex transitive non-Hamiltonian graph. Be sure to use problems 6.103 and 6.104.

**Problem 6.106** Prove or disprove: the generalized Petersen graph $P_5,3$ is Hamiltonian.

**Problem 6.107** Demonstrate, with a non-crossing drawing, that the graph

$$G = \text{Cay}(S_4, \{(1 \ 2 \ 3 \ 4), (1 \ 2)\})$$

is Hamiltonian.

**Problem 6.108** Suppose that $G$ is finite, bipartite, and Hamiltonian. Prove that the halves of the bipartition must be the same size.

**Problem 6.109** For which values of $n$ and $m$ is $C_n \times C_m$ Hamiltonian? Be sure to prove your answer.

**Problem 6.110** Find a Hamilton cycle in the graph above. This graph is called the dodecahedron.

**Problem 6.111** Prove that the dodecahedron, from Problem 6.110, is isomorphic to a generalized Petersen graph.

**Problem 6.112** Prove that the Cartesian product of Hamiltonian graphs is Hamiltonian.

**Problem 6.113** Prove that the girth of a generalized Coxeter graph is at most 12.

**Problem 6.114** Find a generalized Coxeter graph of girth 12. This is difficult without a computer.

**Problem 6.115** Which generalized Coxeter graphs are bipartite?

**Problem 6.116** Use Problem 6.112 to show that an analog to Gray codes exists for strings over the alphabet $\{0, 1, 2, \ldots, k - 1\}$. That is to say prove we can list all length $n$ strings over an alphabet with $k$ letters in an order so that adjacent (and the first and last) strings differ in only one position.

**Problem 6.117** Adopt the representation of $H_n$ given in Problem 6.65 and prove the following:

(i) Changing 0 to 1 and 1 to 0 at a given position in every vertex yields a bijection that is an automorphism.
(ii) Simultaneously permuting the order of the letters in every vertex yields a bijection that is an automorphism.

6.5 Colorings and Drawings

Colorings and drawings of graphs are intimately related topics within graph theory. We will start with a series of definitions that lead up to the chromatic function of a graph.

**Definition 6.50** A coloring of a graph is an assignment of a color to each vertex of the graph. A proper coloring of a graph is a coloring in which adjacent vertices are always assigned distinct colors.

**Definition 6.51** If \( G \) is a finite simple graph then the number of ways to properly color \( G \) in a given number of colors is denoted by the map

\[
\chi_G : \mathbb{N} \rightarrow \mathbb{N}
\]

This map is called the **chromatic function** of the graph.

**Example 6.25** Find the chromatic function \( \chi_{K_n}(m) \).

**Solution:**

Notice that all vertices of \( K_n \) are adjacent and so all the colors must be different. This means there are \( m \) choices for the first color, \( m - 1 \) choices for the second color, and so on until we run out of vertices or colors. Thus for \( m \geq n - 1 \) multiplying these choices together gives us

\[
\chi_{K_n}(m) = m(m - 1) \cdots (m - n + 1)
\]

and for \( m < n - 1 \) we note that we cannot properly color the graph (since we run out of distinct colors before we run out of vertices) and hence \( \chi_{K_n}(m) = 0 \). However since \( m(m - 1) \cdots (m - n + 1) \) evaluates to 0 for \( m \leq n - 1 \), we write the chromatic function as \( \chi_{K_n}(m) = m(m - 1) \cdots (m - n + 1) \).

**Lemma 6.14** The chromatic function of a tree \( T \) on \( n \) vertices is

\[
\chi_T(m) = m \cdot (m - 1)^{n - 1}
\]

**Proof:** This proof is left as an exercise.

**Lemma 6.15** The chromatic function \( \chi_{K_n} = m^n \).

**Proof:**

Since there are no edges, there are \( m \) choices for each color. Multiply. \( \square \)

Relatively few graphs have as simple a counting argument for finding the chromatic function as does the complete graph. Trees are another graph with a very simple formula for their chromatic function, see Problem 6.118. In general computing chromatic functions is tedious. There is a clever trick for creating a recursion that permits us to use a computer to find chromatic functions. As one might expect, this technique requires some definitions.

**Definition 6.52** Contracting an edge of a graph consist of identifying the ends of the edge and fusing the two vertices into a single vertex. This may create multiple edges and, if it does, these are replaced with a single edge. See Example 6.26.

**Example 6.26** Below are examples of graphs with the marked edge \( e \) contracted. The first is \( C_4 \) in which no multiple edges form. The second is \( K_5 \) which forms several multiple edges when \( e \) contracts; the replacement of these multiple edges is shown as an explicit step.

![Graph Example](image)

**Definition 6.53** If \( G \) is a finite simple graph and \( e \in E(G) \) then we define \( G'_e \) to be \( G \) with the edge \( e \) deleted. We define \( G''_e \) to be \( G \) with the edge \( e \) contracted.
6.5. COLORINGS AND DRAWINGS

**Theorem 6.5** Suppose that $G$ is a finite simple graph and that $v \in E(G)$. Then:

$$\chi_G(m) = \chi_{G_1}(m) - \chi_{G_2}(m)$$

This equation is called the **chromatic recursion**.

**Proof:**

Examine all proper colorings of $G_v$. These may be separated into two disjoint sets: those where the ends of $e$ are different colors and those where the ends of $e$ are the same. Each of the former is a proper coloring of $G$. Notice that there is a natural bijection of the latter with proper colorings of $G_v^*$: this graph identifies the ends of $e$ and so agrees with colorings of $G_v^*$ in which the ends of $e$ are the same. This means that the proper colorings of $G_v$ which are proper colorings of $G$ are those that do not correspond to proper colorings of $G_v^*$. The theorem follows. □

**Example 6.27** Find $\chi_{C_4}(m)$.

**Solution:**

Notice that $C_4$ with an edge removed is a path (and hence a tree) on four vertices. $C_4$ with any edge contracted is $K_3$. Applying Theorem 6.5, Example 6.25, and Lemma 6.14 we get:

$$\chi_{C_4}(m) = m(m-1)^3 - m(m-1)(m-2)$$

$$= m(m-1)(m^2 - 3m + 3)$$

**Corollary 6.5** The chromatic function of any finite simple graph is a polynomial.

**Proof:**

The chromatic recursion permits us to decompose the chromatic function of a graph, by repeated applications, into sums and differences of chromatic functions of graphs with no edges. Lemma 6.15 tells us each of these is a positive integral power of the variable. The function is thus polynomial. □

The chromatic function of a graph is also (and more commonly) called the **chromatic polynomial** of a graph.

**Lemma 6.16** If a graph has multiple connected components then the chromatic polynomial of the entire graph is the product of the chromatic polynomials of the individual connected components.

**Proof:**

The number of ways to color each component in $m$ colors is independent of the ways the other components can be colored. The fact that independent choices multiply completes the proof. □

**Definition 6.54** A pendant vertex in a graph is a vertex of degree 1.

**Lemma 6.17** Suppose that $G$ is a graph with a pendant vertex. Suppose $H$ is obtained from $G$ by deleting the pendant vertex and its associated edge. Then

$$\chi_G(m) = (m - 1) \cdot \chi_H(m)$$

**Proof: this proof is an exercise.**

**Lemma 6.18** Suppose that there is an ordering of the vertices of a graph $G$ so that for each vertex $v \in V(G)$ the set of neighbors of $v$ that come before $v$ in the ordering form the vertices of a complete induced subgraph (these neighbors have all possible edges between them). Then the chromatic polynomial of $G$ is a product of monomials, one per vertex, with the monomial for a vertex $v$ being $(m - \nu)$ where $\nu$ is the number of neighbors of $v$ that came before it in the ordering. See Example 6.28.

**Proof:**

Traverse the vertices in order. For each vertex $v$, since its neighbors that come before it in the order are all adjacent, there will be $\nu$ colors already used leaving a choice of $m - \nu$ valid choices for $v$. Multiplying the successive choices yields the polynomial given in the theorem. □.

The preceding lemma is subtle (or confusing) enough to require an example.

**Example 6.28** Consider the graph below. The vertices have been numbered. Verify that the order has the property required by Lemma 6.18, that the neighbors of each vertex that are before it in the ordering are all adjacent. We now traverse these colorings in
order. If \( m \) colors are available there are \( m \) colors available for vertex 1. Since 2 is adjacent to 1, there are \((m - 1)\) colors available for it. Vertices 3-7 will each be adjacent to two vertices that have already been colored when we reach them in the ordering. These two vertices are adjacent, using up 2 colors, and leaving \((m - 2)\) choices of colors for each of these vertices. The chromatic polynomial of this graph is thus

\[
m(m - 1)(m - 2)^5
\]

**Definition 6.55** Suppose that \( G \) and \( H \) are graphs and that \( v \in V(G) \) and \( u \in H(G) \). The the join of \( G \) and \( H \) at \( v \) and \( u \) is the result of taking a copy of \( G \), a copy of \( H \), but identifying \( u \) and \( v \) to form a single vertex. See Example 6.29.

**Example 6.29** Below is an example of joining two copies of \( C_3 \) to make a five-vertex graph called the bow tie.

\[
\begin{align*}
\text{v} &\quad \text{u} \\
1 &\quad 2 \\
3 &\quad 4 \\
5 &\quad 6 \\
7 &
\end{align*}
\]

**Lemma 6.19** Suppose that \( G \) is the join of two graphs \( H \) and \( K \). Then

\[
\chi_G(m) = \frac{1}{m} \chi_H(m) \cdot \chi_K(m)
\]

**Proof:** this proof is an exercise.

**Definition 6.56** The chromatic number of a graph is the smallest number of colors needed to properly color the graph. The chromatic number of a graph \( G \) is denoted \( \chi(G) \).

**Theorem 6.6** Suppose that the highest degree of \( G \) is at most \( \delta \). Then \( \chi(G) \leq \delta + 1 \).

**Proof:**

Place the vertices of \( G \) in some order and also number a list of colors \( c_1, c_2, c_3, \ldots \). Considering the vertices of \( G \) in order, color each with the smallest color not already assigned to its neighbors. Since each vertex has at most \( \delta \) neighbors no color larger that \( c_{\delta + 1} \) can be required. We thus may color \( G \) properly in \( \delta + 1 \) colors. \( \Box \)

Brook’s theorem, which we do not prove, states that the only graphs for which the chromatic number exceeds the maximum degree are the complete graph and odd cycles. For all other graphs the chromatic number is at most the degree. Notice that the chromatic number if also the smallest \( m \) for which the chromatic polynomial is not zero. We now begin the connection of chromatic theory with drawing.

**Definition 6.57** A graph is said to be planar if it can be drawn in the plane with no two edges crossing.

**Definition 6.58** A drawing of a graph is rectilinear if all the edges are drawn as straight line segments.

We now know enough to state one form of the most famous theorems in graph theory. This theorem is offered without proof (the proof is very long and requires the examination of in excess of 1400 graphs covering cases of the theorem).

**Theorem 6.7 (The Four Color Theorem)** The chromatic number of a planar graph is at most four.

The four color theorem started as a conjecture about (cartographic) maps. The connection between cartographic maps and combinatorial graphs is not a difficult one.

**Definition 6.59** When a graph is drawn in the plane without any of its edges crossing one another, then it divides the plane into regions, including an infinite region outside of the graph. In a cartographic map these are called countries. In a non-crossing drawing of a planar graph these regions are called the faces of the graph.

**Definition 6.60** Constructing the planar dual of a graph requires a drawing of the graph in the plane. The vertices of the planar dual are the faces of the
drawing of the original graph. Edges are pairs of faces that share an edge of the original graph as a boundary.

**Example 6.30** The planar dual of the cube. Below is the drawing of the cube, \( H_3 \), done with black vertices. The six four-cycles in \( H_3 \) each surround a four-sided face, including the outer face consisting of the exterior of the cube. Within each face (including the outer face) is a white vertex. Another graph is drawn with dotted lines for edges. Two white vertices are connected if their corresponding faces have an edge in common. The second graph is the planar dual of this drawing of the cube.

There is no reason that the planar dual of a graph must be a simple graph. It is often the case that the planar dual contains multiple edges and loops.

**Definition 6.61** The number of edges that cross in a drawing of a graph are called the crossing number of the drawing. It is always possible to draw edges so that at most any two edges cross at any one point. A crossing number is always computed without triple-or-larger crossings.

**Definition 6.62** The smallest crossing number of any drawing of a graph is the crossing number of the graph.

Notice that being planar is the same as having crossing number zero.

### 6.5.1 The Plane and the Sphere

One of the awkward features of a drawing of a planar graph is the outer face. The drawing of a planar graph breaks the plane into some number of finite regions and one infinite region. This infinite region is connected to the problem of finding a way of representing the surface of a sphere on a flat map which is called the projection problem.

**Theorem 6.8 Polar Projection** A graph may be drawn without crossings on the surface of a sphere if and only if it may be drawn without crossings in the plane.

**Proof:**

Place a sphere so that it is tangent to the Cartesian plane at the origin. Call the point of tangency the south pole and the point on the opposite side of the sphere the north pole. Any line through the north pole that intersects the sphere in another point also intersects the plane in a single point. Notice that all points in the plane lie on lines through the north pole that intersect the sphere and, since two points deter-mine a line, these lines are unique. Likewise every point on the sphere that is not the north pole lies on a line through the north pole that intersects the plane. If we make a correspondence between points on the plane and points on the sphere that lie on these lines through the north pole then that correspondence is a bijection of the plane with the sphere except for the north pole. This bijection is called the polar projection of the sphere onto the plane.

Notice that a drawing of a graph may be translated on the surface of the sphere so that it does not include the north pole. Rigid translation does not create or destroy crossings. This means that the polar projection, with a properly chosen north pole, can be used to transfer a drawing back and forth between the sphere and the plane. By "properly chosen north pole" we simply mean that the north pole lies in the interior of a face for the non-crossing drawing of the graph on the sphere. The polar projection, in either direction, must take crossings to crossings and so preserve the crossing number of the drawing. □

When we transfer a planar drawing of a graph to the sphere the infinite part of the plane becomes a finite part of the sphere that has the north pole in its interior. If we draw a planar graph on the surface of a sphere without crossings and then flatten the faces into polygons the result is a polyhedron. We offer
without proof the following fact. There are five polyhedra that can be constructed so that all the faces are identical regular polygons. They are called the Platonic solids. Each has a corresponding planar graph whose drawing it can be created from. They are:

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>Four triangular faces, four vertices, six edges. The graph is $K_4$.</td>
</tr>
<tr>
<td>Cube</td>
<td>Six square faces, eight vertices, twelve edges. The graph is $H_3$.</td>
</tr>
<tr>
<td>Octahedron</td>
<td>Six triangular faces, six vertices, twelve edges. The graph is called the octahedron.</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>Twelve pentagonal faces, twenty vertices, thirty edges. The graph is the dodecahedron.</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>Twenty triangular faces, twelve vertices, thirty edges. The graph is called the icosahedron.</td>
</tr>
</tbody>
</table>

There is a curious connection between the graphs of the Platonic solids via their planar duals. The tetrahedron is isomorphic to its own planar dual. The cube and octahedron are each isomorphic to the planar dual of the other. The same is true of the dodecahedron and icosahedron. You are asked to verify most of this in the exercises.

**Definition 6.63** A graph that is isomorphic to its own planar dual is said to be self-dual.

**Definition 6.64** A wheel graph is constructed from $C_n$ by adding a single new vertex that is adjacent to all others. The wheel graph is denoted $W_n$.

**Example 6.31** Wheel graphs Shown below are the wheel graphs $W_4$, $W_5$, $W_6$

![Wheel graphs](attachment:wheel_graphs.png)

**Lemma 6.20** The wheel graphs are self-dual.

Proof: this proof is an exercise.

The planar dual of a graph depends on the drawing of the graph. In spite of this we have at several times in this section referred to "the" planar dual of a graph. In fact the planar dual of all the Platonic solids and wheel graphs, for example, are unique. To be more precise the planar dual of any two planar drawing of these graphs are isomorphic. We now give definitions and a theorem, without proof, that justify this use of the definite article.

**Definition 6.65** Suppose the $G$ is a connected graph. A vertex cut of $G$ is any subset of $V(G)$ that will disconnect the graph if the vertices of the subset (and their associated edges) are removed.

**Definition 6.66** The connectivity or vertex connectivity of a graph is the size of a minimal vertex cut. A graph with a vertex connectivity of $k$ is said to be $k$-connected. Two-connected graphs are said to be biconnected while three-connected graphs are said to be tri-connected. The connectivity of a graph $G$ is denoted by $\kappa(G)$.

The following theorem was proved by Hassler Whitney in 1932. His theorem was phrased in terms of equivalence classes of embeddings of graphs in the sphere, but the following is a direct consequence.

**Theorem 6.69** A planar 3-connected graph has a unique planar dual.

Planar duals need not be simple graphs, but it turns out that the planar dual of a 3-connected graph is a simple graph. Computing a graph’s connectivity is a bit tricky without the following results.

**Lemma 6.21** The vertex connectivity of a graph is at most the smallest degree of any vertex of the graph.

Proof: this proof is an exercise.

**Definition 6.67** Let $G$ be a graph and let $u, v \in V(G)$. Two paths with ends at $u$ and $v$ are independent if the only vertices they have in common are $u$ and $v$.

The following theorem is due to Menger. It is offered here without proof but is a special case of the max-flow min-cut theorem that appears in the section on graph algorithms.
6.5. COLORINGS AND DRAWINGS

**Theorem 6.10** The smallest number of independent paths between any two vertices in a graph is equal to the vertex connectivity of the graph.

**Example 6.32** Prove that the cube is 3-connected.

![Cube Diagram]

**Solution:**

Recall that the cube is vertex transitive and refer to the labeled cube above. Vertex transitivity means that we need only verify that there are three independent paths from one vertex to all others. In addition, the stabilizer Aut(H₃)ₐ has three orbits: \{b, d, e\}, \{c, f, h\}, and \{g\}. To see this note there is an automorphism, fixing \( a \), of order three that consists of rotating the (geometric) cube \( \frac{2\pi}{3} \) about the diagonal from \( a \) to \( g \). This automorphism realizes these orbits. Since automorphisms preserve paths, this means that we need only verify that there are three (and at most three) mutually independent paths from \( a \) to \( b \), \( c \), and \( g \). For \( a \) and \( b \) one choice of these paths are: \( ab, ae, fb \) and \( ad, efb \). In both cases we exhaust all neighbors of \( a \) and hence there are no other paths which can be mutually independent. Similarly for \( a \) and \( e \) possible paths are \( ace, adf, \) and for \( a \) and \( g \) possible paths are \( abg, aef, \) and \( adh \).

Example 6.32 relied heavily on the fact that \( H₃ \) is highly symmetric. We will develop other tools for demonstrating connectivity in later sections. We now turn to an interesting regularity of planar drawings of graphs discovered by Euler.

**Theorem 6.11 Euler’s Formula** In a planar drawing of a finite connected graph \( G \) with \( v \) vertices, \( e \) edges, and \( f \) faces we have that

\[
v - e + f = 2
\]

**Proof:**

Notice that any planar drawing of a tree does not separate the plane and so has one face. Recall that a tree has one less edges than vertices. A drawing of a tree thus has \( v \) vertices, \( v - 1 \) edges, and 1 face. Checking Euler’s formula:

\[
v - (v - 1) + 1 = v - v + 2 = 2
\]

and so the theorem holds for trees.

Consider now a planar drawing of the graph \( G \). Starting with any vertex of \( G \) construct a tree that is a subgraph of \( G \) as follows. Repeatedly add a new vertex that is adjacent to some vertex already in the tree together with the corresponding edge. Since \( G \) is connected there are paths between any two vertices and so such additions may be performed until we have a tree-subgraph that includes all of \( V(G) \). We may complete \( G \) by adding edges to the tree. Each time we add an edge, increasing \( e \) by one, we also divide a face of the current drawing into two faces increasing \( f \) by one. Thus adding back all the remaining edges of \( G \) keeps the total \( v - e + f \) constant and the theorem follows.

The tree constructed in the proof of Theorem 6.11 is a special type of tree that includes all the vertices of the graph. Trees of this type are quite handy and exist in any connected graph.

**Definition 6.68** If \( G \) is a graph then a spanning tree of \( G \) is a subgraph with vertex set \( V(G) \) that is, itself, a tree.

**Corollary 6.6** A graph \( G \) has a spanning tree if and only if it is connected.

**Proof:**

The construction of a spanning tree for a connected graph is given in the proof of Theorem 6.11. A graph that is not connected cannot have a spanning tree because that tree must contain paths connecting any two vertices and so forms a witness that the graph is connected.

We next examine a theorem of Tutte about drawings of planar graphs. This theorem is remarkable easy to implement as a computer program and provides an easy source of non-crossing rectilinear drawings. We offer the theorem without proof.
Definition 6.69 A non-separating cycle in a connected graph is a cycle with the property that deleting the cycle from the graph yields a new graph with a single connected component.

Theorem 6.12 Tutte
Let $G$ be a 3-connected planar graph and let $C$ be a non-separating cycle of $G$. Fix the positions of the vertices of $C$ as a convex polygon in the plane. For all other vertices solve the system of linear equations defined by the rule “my position is the average of my neighbors positions” where averaging is done independently in $x$ and $y$. Then if all edges are drawn as line segments the resulting drawing is non-crossing.

Example 6.33 The following pair of drawings of $P_{10,2}$ and $P_{12,2}$ were done using Theorem 6.12.

The linear system specified in Theorem 6.12 is very well behaved. If we call the vertices of the non-separating polygon the fixed vertices and the other mobile vertices then iteratively selecting a mobile vertex at random and moving it to the average position of its neighbors will rapidly converge to a rectilinear, planar drawing.

Definition 6.70 A Fullerene graph is a 3-connected planar graph in which all faces are pentagons or hexagons.

Proposition 6.1 The number of pentagons in a Fullerene graph must be exactly 12.

Proof: this proof is left as an exercise.

Problems


Problem 6.119 Compute the chromatic polynomial of $C_9$. Hint: compute several examples and look for a pattern.

Problem 6.120 Prove Lemma 6.17.

Problem 6.121 Compute the chromatic polynomial of $K_{3,3}$. You can apply the chromatic recursion so as to exploit both Lemma 6.17 and Lemma 6.18.

Problem 6.122 Read Lemma 6.18 and review Example 6.28. Construct a graph with chromatic polynomial

$$n(n-1)^2(n-2)^3(n-3)(n-4)^2$$

Problem 6.123 Find an ordering of the vertices of the bowtie graph from Example 6.29 that permits you to apply Lemma 6.18 to compute its chromatic polynomial.

Problem 6.124 Suppose that $n_0, n_1, \ldots, n_k$ are a finite sequence of positive integers. Prove that there is a graph that has the chromatic polynomial

$$\prod_{i=0}^{k} (m-i)^{n_i}$$


Problem 6.126 Prove that the coefficients of the chromatic polynomial of a graph must alternate in sign.

Problem 6.127 Prove that the chromatic number of a cycle is 3 if the cycle is of odd length and 2 if it is of even length.

Problem 6.128 Prove that the chromatic number of a connected graph, with at least two vertices, is two if and only if the graph is bipartite.

Problem 6.129 Compute the chromatic number of the Petersen graph.

Problem 6.130 The algorithm used to prove Theorem 6.6 is an example of a greedy algorithm. Find orderings of the vertices of $H_3$ that cause this algorithm to use 2, 3, and 4 colors.

Problem 6.131 Read problem 6.130. The largest number of colors that the algorithm used in Theorem 6.6 can produce, for any ordering of the vertices, is called the Grundy number of the graph. Find the Grundy number of $H_4$ and of $K_{3,3}$. 
6.5. COLORINGS AND DRAWINGS

Problem 6.132 Read the proof of Theorem 6.6. Prove that an ordering of the vertices exists, for any graph \( G \), that will cause this algorithm to use the smallest possible number of colors, \( \chi(G) \).

Problem 6.133 Prove that \( K_n \) is planar iff \( n \leq 4 \).

Problem 6.134 Prove that \( K_{n,m} \) is planar iff the minimum of \( n \) and \( m \) is at most two.

Problem 6.135 Prove that the prism \( P_n \) is planar.

Problem 6.136 Prove that the generalized Petersen graph \( P_{2n,2} \) is planar.

Problem 6.137 Prove that a cubic graph is planar iff its triangulation is planar.

Problem 6.138 Compute the crossing number of \( K_5 \).

Problem 6.139 Compute the crossing number of \( K_{3,3} \).

Problem 6.140 Compute the crossing number of the Petersen graph.

Problem 6.141 Compute the maximum crossing number of any rectilinear drawing of \( K_n \).

Problem 6.142 Make a drawing that verifies that the tetrahedron is self-dual.

Problem 6.143 Make a drawing that verifies that \( H_3 \) and the octahedron, drawn above, are mutual planar duals.

Problem 6.144 Make a rectilinear, non-crossing drawing of \( H_3 \).

Problem 6.145 Compute the size of the automorphism group of the octahedron. Hint: consider the compliment.

Problem 6.146 Make a planar drawing of the icosahedron. Hint: there are planar drawings of the dodecahedron earlier in this chapter.

Problem 6.147 Prove that all the Platonic solids are vertex transitive.

Problem 6.148 Find representations of as many of the Platonic solids as Cayley graphs as you can.

Problem 6.149 It is possible to tile the plane in regular triangles, squares, and hexagons. No other single regular polygon will tile the plane. If we treat places where line segments meet as vertices and line segments as edges, each of these regular tilings may be treated as infinite graphs. Compute the planar dual of each of these graphs.

Problem 6.150 Prove Lemma 6.20.

Problem 6.151 Find a finite, self-dual planar graph that is not isomorphic to a wheel graph.

Problem 6.152 Define \( W_n^6 \) to be a graph, derived as the wheel graph is, from \( C_n \). Rather than adjoin one new vertex to every vertex of the cycle we adjoin two such vertices. The Octahedron would be one of these graphs: \( W_4^6 \). Prove that these graphs, for \( n \geq 3 \), are planar.

Problem 6.153 Prove that the graphs \( W_n^6 \) defined in Problem 6.152 are 3-connected and hence have a unique planar dual.

Problem 6.154 Compute and name the planar dual of the graphs \( W_n^6 \) defined in Problem 6.152.


Problem 6.156 Prove that \( K_{3,3} \) is 3-connected. Hint: it is vertex transitive.

Problem 6.157 Prove that the Petersen graph is 3-connected.

Problem 6.158 Prove that \( H_n \) is n-connected.
Problem 6.159 Find a 2-connected planar graph together with two drawings that demonstrate non-isomorphic planar duals. Hint: these duals will probably not be simple graphs.

Problem 6.160 Verify Theorem 6.11 for each of the Platonic solids.

Problem 6.161 State and prove the natural generalization of Euler’s formula (Theorem 6.11) for planar drawings of graphs with multiple connected components.

Problem 6.162 Apply Theorem 6.12 to a non-separating 3-cycle of the Octahedron and carefully render the resulting drawing.


Problem 6.164 Find a Fulleren graph with no hexagons.

Problem 6.165 Prove that if \( n \geq 0 \) is a whole number then there is a Fulleren graph with more than \( n \) hexagons.

Problem 6.166 Suppose we have a 3-connected planar graph in with degree sequence \( 5^a6^b \). Prove that \( n = 12 \).

6.6 Distances in Graphs

In mathematics measures of distance are called metrics. The best known metric is the Euclidean metric on the Cartesian plane that measures the distance from the point \( P = (x, y) \) to the point \( Q = (a, b) \) as

\[
D(P, Q) = \sqrt{(x-a)^2 + (y-b)^2}
\]

In this section we will introduce metric spaces and prove that it is easy to place a metric space structure on the vertices of a graph.

Definition 6.71 A metric space is a set \( M \) together with a function

\[
d : M \times M \to \mathbb{R}
\]

that obeys the three axioms:

1. \( d(x, x) = 0 \), \( \forall x \in M \)
2. \( d(x, y) > 0 \), \( \forall x, y \in M \) unless \( x = y \), and
3. \( d(x, y) + d(y, z) \geq d(x, z), \forall x, y, z \in M \)

The function \( d \) is said to be a metric on \( M \).

Definition 6.72 The length of a path \( P \) is the number of edges in \( P \).

Lemma 6.22 Let \( G \) be a graph and let

\[
d_p : V(G) \times V(G) \to \mathbb{N}
\]

be defined by \( d_p(u, v) \) is the length of the shortest path with ends \( u \) and \( v \). If no such path exists because \( u \) and \( v \) are in distinct connected components the distance is infinite. Then \( d_p \) is a metric on \( V(G) \).

Proof:

Note that \( \mathbb{N} \subset \mathbb{R} \) and so the function \( d_p \) has the correct domain and range. We now check the metric space axioms. If \( v \) is a vertex then the path \( v \) is a path from \( v \) to itself of length zero and so \( d_p(v, v) = 0 \) and we have (i). If \( v \neq u \) then there can be no path of length zero from \( v \) to \( u \) and we have \( d_p(u, v) > 0 \). It remains to check the triangle inequality. Let \( x, y, z \in V(G) \). If all three vertices are not in the same connected component then two sides of the triangle must be infinite and (iii) holds. If all three vertices are in the same connected component then let \( P \) be a path from \( x \) to \( y \) of length \( d_p(x, y) \) and let \( Q \) be a path of length \( d_p(y, z) \) from \( y \) to \( z \). If these paths share no internal vertices then following one then the other provides a witness that \( d_p(x, y) + d_p(y, z) = d_p(x, z) \). If these paths share internal vertices, then we may follow \( P \) part way from \( x \) to \( y \) until it intersects \( Q \) and then follow the remainder of \( Q \) to \( z \) showing that \( d_p(x, y) + d_p(y, z) > d_p(x, z) \). In either case (iii) holds.

To deal with the fact that graphs can have multiple connected components we call the distance infinite between nodes in distinct connected components and treat infinity as a number in these cases. The means that the path metric is a metric using the extended real numbers (the real numbers plus infinity). Under the convention that infinity exceeds any other real
6.6. DISTANCES IN GRAPHS

number in magnitude and that infinite plus anything is infinity, this preserves the properties needed to have a metric, save that the range of the metric is the extended real numbers. When a graph is connected there is no problem, the path metric is a standard metric.

Definition 6.73 The function \( d_p \) defined in Lemma 6.22 is called the path metric for graphs. When we speak of the distance between vertices without other qualification we are speaking of the path metric. Along with the convention that the distance between vertices in distinct connected components is infinite. Though this is not strictly speaking a metric.

Now that we have a notion of distance on graphs we can make a number of definitions that yield both new graphs and new ways of understanding the structure of graphs.

Definition 6.74 If \( G \) is a graph and \( n \) is a positive whole number then the \( n \)th power of \( G \), denoted \( G^n \), is a graph for which

\[
V(G^n) = V(G)
\]

and

\[
E(G^n) = \{ \{u, v\} : d_p(u, v) \leq n \text{ in } G \}
\]

The power of a graph adds edges to a graph between vertices that are less than some distance apart in the original graph.

Example 6.34 Shown below are drawings of powers of the 8-cycle, \( C_8 = C_8^1, C_8^2, \) and \( C_8^3 \).

![Diagram of powers of 8-cycle](image)

Definition 6.75 The eccentricity of a vertex \( v \) is the maximum distance any other vertex is from that vertex. It is denoted \( ecc(v) \).

There is a simple method for computing the eccentricity of a vertex in a connected graph that is not difficult to implement on a computer and which can be done with pencil and paper for graphs that are not too large. Note that all vertices in a graph that is not connected have infinite eccentricity.

Algorithm 6.1 Computing Eccentricity

Input: A connected finite graph \( G \), a vertex \( x \in V(G) \)
Output: \( ecc(v) \)
Details:
1) Number \( v \) with zero
2) While unnumbered vertices remain
   3) Number each unnumbered vertex
      that is next to a numbered vertex
      with \( k + 1 \) where \( k \) is the smallest
      number on an adjacent vertex
4) End while
The largest numeric label is \( ecc(v) \)

Example 6.35 Apply Algorithm 6.1 to the Petersen graph to compute the eccentricity of a vertex.

Solution:

![Petersen graph with eccentricity values](image)

So we see that the eccentricity of the topmost vertex in the above drawing is zero. The fact that the Petersen graph is vertex transitive means that all vertices have eccentricity 2.

Definition 6.76 The diameter of a graph is the maximum eccentricity of any vertex in the graph. The radius of a graph is the minimum eccentricity of any vertex in the graph.

Notice that Example 6.35 demonstrates that the radius and diameter of the Petersen graph are both 2, since this is the eccentricity of each vertex.
**Definition 6.77** The periphery of a graph is the induced subgraph on the vertices of maximum eccentricity. The center of a graph is the induced subgraph on the vertices of minimal eccentricity. The annulus of a graph is the induced subgraph on all vertices not in the center or periphery.

**Definition 6.78** Vertices in the periphery of a graph as said to be peripheral. Vertices in the center of a graph are said to be central.

**Example 6.36** Shown below is a tree with the vertices in the periphery colored white, those in the center large and black, and the annular vertices small and black.

**Lemma 6.23** The diameter of $C_n$ is $\lceil n/2 \rceil$.

Proof:

Recall that $C_n$ is vertex transitive and so we may start at any vertex to find a witness to the diameter. Apply Algorithm 6.1. The chosen vertex is numbered zero; after that pairs of vertices are numbered 1, 2, 3, until the last vertex is numbered $\frac{n}{2}$ if $n$ is even or a last pair of vertices are numbered $\frac{n-1}{2}$ if $n$ is odd. This makes the largest numeric label $\lceil \frac{n}{2} \rceil$ and the lemma follows. □

We have been using the following Lemma implicitly and now need to state it explicitly.

**Lemma 6.24** Graph isomorphisms (and hence automorphisms) preserve distances in graphs.

Proof:

Lemma 6.6 says that isomorphisms preserve the presence of a particular subgraph. For any distance in a graph there is a path of minimal length between the two vertices which is a witness to the distance. Lemma 6.6 says that isomorphism preserves this witness; since the inverse of an isomorphism is also an isomorphism it cannot create a shorter path and hence reduce the distance. The lemma follows. □

**Corollary 6.7** For a finite simple graph $G$, let $A = \text{Aut}(G)$. Then for $v \in V(G)$ the stabilizer $A_v$ of $v$ has at least $\text{ecc}(v) + 1$ orbits and acts so as to preserve the distance of other vertices from $v$.

Proof:

That $A_v$ preserves distance from $v$ is a restatement of Lemma 6.24. There is at least one vertex, other that $v$, at distance 1, 2, ..., $\text{ecc}(v)$ from $v$. Each of these is in its own orbit of $A_v$ as is $v$ itself, explicitly locating $\text{ecc}(v) + 1$ orbits. □

### 6.6.1 An Application

It is common in mathematics to have a measure of how different two objects are without being able to prove that that measure is actually a metric, satisfying the three metric space axioms. In many cases, graph theory can help us make that jump.

**Definition 6.79** Suppose that $s$ and $t$ are character strings. Then the edit distance, $d_e(s,t)$ between $s$ and $t$ is the smallest number of single character insertions, substitutions, or deletions that can transform $s$ into $t$.

**Example 6.37** Notice that $d_e(BAT, STATE) = 3$ via:

<table>
<thead>
<tr>
<th>Original</th>
<th>Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>BAT</td>
<td>substitute T for B</td>
</tr>
<tr>
<td>STAT</td>
<td>insert S on the left</td>
</tr>
<tr>
<td>STATE</td>
<td>insert E on the right</td>
</tr>
</tbody>
</table>

**Lemma 6.25** The function $d_e$ is a metric on the space of finite strings over a given alphabet $A$.

Proof:
6.6. DISTANCES IN GRAPHS

Let $G(V,E)$ be the graph in which $V(G)$ is the set of all finite strings over $A$. Let $E(G)$ be all pairs of strings that differ by a single insertion, deletion, or substitution. Then the value of the path metric in this graph is identical to the edit distance between strings. This means that the edit distance obeys the metric axioms when the path metric does. The only thing we need to still show is that the graph is connected. Consider any two finite strings $s$ and $t$, there is clearly a finite path between $s$ and the null string, simply delete each character in the string, and similarly there is a finite path between $t$ and the null string, thus there is a path between $s$ and $t$, hence the graph is connected. □

A very large number of distance measures can be shown to be metrics by showing that they are path metrics in a well chosen graph.

6.6.2 An Unsolved Problem

In problem 6.11 we defined a strange graph. The vertices of this graph are the Cartesian plane and edges are pairs of vertices at distance one. This graph is the unit graph on the plane. We will denote this graph by $U$. This graph is interesting because its chromatic number $\chi(U)$ is finite and its value is unknown. Computing the chromatic number of $U$ is known as the Hadwiger-Nelson problem. We will review the known bounds. The next lemma gives a flavor of the type of proof used to deal with $U$.

**Lemma 6.26** $\chi(U) \geq 3$.

**Proof:**

An equilateral triangle with side length one yields a copy of $K_3$ as a subgraph of $U$ that requires three colors. □

**Definition 6.80** Graphs that are isomorphic to subgraphs of $U$ are called unit graphs. Finite unit graphs are called gadgets. A gadget with a chromatic number of $k$ is called a $k$-gadget.

$K_3$ is an example of a 3-gadget. The graph used in the proof of lemma 6.27 is an example of a 4-gadget and is believed to be the smallest 4 gadget. No 5-gadget is known.

The next result is a little startling.

**Lemma 6.28** $\chi(U) \leq 7$.

**Proof:**

Scale the diagram below, which uses a repeating motif of hexagons with seven distinct colors represented as numbers (ignore the black lines), so that the main diagonal of each hexagon is slightly less than one. Continue the diagram to cover the plane. Then any two hexagons that are the same color are more than a unit apart at their closest approach; points within a given hexagon are less than a unit apart. The diagram thus extends to an explicit, proper seven-coloring of $U$. □
Lemma 6.27 and 6.28 represent, at the time of this writing, the limits of our knowledge on \( \chi(U) \); the number is between 4 and 7. We offer, without proof, the fact that the chromatic number of \( U \) is realized in one of its finite subgraphs. Another interesting fact that we offer without proof is that the induced subgraph of \( U \) in which vertices are points in which both coordinates are rational is bipartite and so has a chromatic number of two.

**Problems**

**Problem 6.167** Verify that the Euclidean metric in the plane obeys the three metric space axioms.

**Problem 6.168** Adopt the representation of \( H_n \) given in Problem 6.65. Prove that the distance between any two vertices is the number of positions in which the strings corresponding to those vertices disagree.

**Problem 6.169** Find \( G \) so that \( G^2 \) is the graph drawn below.

This is an example of finding a square root of a graph.

**Problem 6.170** Show that every power of the cycle \( C_n \) is a difference graph and find a difference graph that is not a power of a cycle.

**Problem 6.171** Prove that every power of a Cayley graph is a Cayley graph.

**Problem 6.172** Recall that \( P_n \) is a path with \( n \) vertices. Prove that \( P_n^2 \) is planar.

**Problem 6.173** Find all finite simple graphs for which \( G^2 = G \).

**Problem 6.174** Let \( G \) be a finite simple graph and consider the sequence

\[
(G^1, G^2, G^3, \ldots, G^n, \ldots)
\]

of graphs. Demonstrate that the sequence becomes constant after some number of steps and determine on which step the final graph in the sequence appears.

**Problem 6.175** Suppose that \( G \) is a connected finite simple graph on \( n \geq 3 \) vertices. Prove that the girth of \( G^2 \) is 3.

**Problem 6.176** Prove that the center of a finite tree is isomorphic to either \( K_1 \) or \( K_2 \).

**Problem 6.177** Prove that the center of a vertex transitive graph is the entire graph.

**Problem 6.178** Find a spanning trees for the Petersen graph of maximal and minimal diameter.

**Problem 6.179** Compute the diameter of \( H_n \).

**Problem 6.180** Use Algorithm 6.1 to compute the eccentricity of \( P_n \). Use a planar drawing of \( P_n \).

**Problem 6.181** Compute the diameter of \( P_n \).
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**Problem 6.182** Draw a copy of the graph above and label each vertex with its eccentricity.

**Problem 6.183** Construct a tree on eight vertices which has the smallest possible number of peripheral vertices.

**Problem 6.184** Construct a tree on eight vertices which has the largest possible number of peripheral vertices.

**Problem 6.185** Prove that all peripheral vertices in a tree are leaves. Recall that the leaves of a tree are the vertices of degree one.

**Problem 6.186** Read problem 6.185. Find a tree with at least one leaf that is not peripheral.

**Problem 6.187** Find a graph in which the center of the graph is the entire graph but the graph is not vertex transitive.

**Problem 6.188** Compute (probably by giving a clear sequence of drawings) the set of all trees with the following properties:

(i) The degree sequence is $3^{n-2}1^2$ for some $n$.

(ii) The center consists of a single vertex.

(iii) All leaves are peripheral.

**Problem 6.189** Prove or disprove: one of the trees from Problem 6.188 appears as a spanning tree of the Petersen graph.

**Problem 6.190** Compute (probably by giving a clear sequence of drawings) the set of all trees with the following properties:

(i) The degree sequence is $3^{n-2}1^2$ for some $n$.

(ii) The center consists of a copy of $K_2$.

(iii) All leaves are peripheral.

**Problem 6.191** Explicitly determine the orbit of the stabilizer in $\text{Aut}(H_3)$ of any one vertex; check the distance of the members of each orbit from the fixed vertex.

**Problem 6.192** Lemma 6.25 proves the edit metric is a metric by constructing the graph. For the alphabet $\mathcal{A} = \{0, 1\}$ construct and draw the induced subgraph on strings of length at most three.

**Problem 6.193** This problem concerns the Cartesian plane. Define a function $d_u(P, Q)$ from pairs of points $P$ and $Q$ in the Cartesian plane to the integers as follows. Find a smallest sequence of unit vectors (it need not be unique) that start at $P$ and end at $Q$. $d_u(P, Q)$ is the length of such a sequence. Prove that $d_u(P, Q)$ is a metric.

**Problem 6.194** Prove $C_n$ is a unit graph.

**Problem 6.195** Prove that any tree is a unit graph.

**Problem 6.196** Prove that the Petersen graph is a unit graph.

**Problem 6.197** Prove that the prism of a gadget is a gadget. Use this to prove $H_n$ is a gadget.

**Problem 6.198** Find a 4-gadget of your own (not the one that appears in the text). Do not use too many vertices.

**Problem 6.199** Construct a graph that is not a unit graph with as few vertices as possible.

**Problem 6.200** In Problem 6.149 three infinite graphs are defined. Compute the chromatic number of each of these.

**Problem 6.201** Suppose we define the unit graph on three-space, in analogy to the unit graph on the plane. Find a finite subgraph that is a witness that this graph has a chromatic number of at least five.

### 6.7 Topological Graph Theory

We have already begun the study of topological graph theory with the notion of a planar graph and the planar dual of a graph. In this section we will look at the theory of forbidden subgraphs for determining planarity and embeddings of graphs in more general surfaces than the plane and sphere.
Definition 6.81 A subdivision of a graph \( G \) results from inserting new vertices into edges:

\[ \begin{array}{cc}
  & u \\
  & \downarrow \\
  w & \downarrow \\
  & v \\
\end{array} \]

Here \( u \) and \( v \) are the original vertices and \( w \) is inserted. Subdivision removes an existing edges and creates two new ones. One graph is said to be a subdivision of another if the second graph is obtained by any number of subdivisions of edges of the first.

Definition 6.82 Reversing a subdivision is called smoothing.

Definition 6.83 Two graphs \( G \) and \( H \) are homeomorphic if there is are subdivisions \( G' \) of \( G \) and \( H' \) of \( H \) so that \( G' \cong H' \).

Since smoothing and subdivision are opposites of one another it is possible to demonstrate homeomorphism via smoothing or subdivision. When demonstrating two graphs are homeomorphic, do your best to choose whichever approach is easier.

The following theorem, Kuratowski’s Theorem, is an exact characterization of planarity and is offered without proof.

Theorem 6.13 A graph is planar iff it contains no subgraph homeomorphic to either \( K_5 \) or \( K_{3,3} \).

Example 6.38 Demonstrate, using Kuratowski’s theorem, that the Petersen graph is not planar.

Solution:

In Problem 6.63 we saw that the graph on the left above is a drawing of the Petersen graph. Deleting the vertex in the middle yields the subgraph and then three acts of smoothing yield \( K_{3,3} \). This means the middle graph is a subdivision of \( K_{3,3} \) and so a subgraph of the Petersen graph is homeomorphic to \( K_{3,3} \), satisfying Kuratowski’s criterion for non-planarity.

Kuratowski’s theorem uses homeomorphism as a way of understanding when a graph “contains a copy” of \( K_{3,3} \) or \( K_5 \), the two graphs that are forbidden to appear “as a copy” in any planar graph. There is a second way to approach the same intuitive concept.

Definition 6.84 We say a graph \( H \) is an edge contraction of a graph \( G \) if we obtain \( H \) from \( G \) as follows. Pick an edge or \( G \), delete it, and then fuse its former ends into a single vertex. Any multiple edges that are created are replaced by single edges.

Definition 6.85 We say that a graph \( K \) is a minor of a graph \( G \) if \( K \) is isomorphic to a graph obtained from \( G \) by a (possibly empty) sequence of edge contractions.

The following theorem, Wagner’s Theorem, is an alternative to Kuratowski’s theorem for determining planarity.

Theorem 6.14 A graph is planar iff it does not have either \( K_5 \) or \( K_{3,3} \) as a minor.

Example 6.39 Using Wagner’s Theorem, prove that the Petersen graph is not planar.

Solution:

Note that the Petersen graph may be thought of as a pentagon and five-pointed star connected by spokes. Contract all the spokes and obtain \( K_5 \).
6.7. TOPOLOGICAL GRAPH THEORY

One of the original motivations for studying the crossing number of graphs is laying out components on a printed circuit board. An example of a circuit board is shown above, with the conductors shown in purple ink. In an electrical circuit we may think of the components (transistors, capacitors, resistors, inductors, diodes, and integrated circuits) as vertices in a graph. This graph may have other vertices, like places where wires join. The conductors joining components are edges of the graph. A printed circuit board is a non-conducting board with a conductor, typically copper, printed where electrical connections must go. Any point where two edges must cross without touching is a place where a jumper wire is required. Since a jumper wire is an added manufacturing step, minimizing the number of crossings in the drawing of a graph is a step in efficiently manufacturing electronic circuits. This application motivates the following definition.

Definition 6.86 Suppose that we draw a graph using \( k \) colors for the edges. The thickness of a graph is the smallest number of colors in which a drawing exists so that all pairs of edges that cross are different colors. We denote the thickness of a graph \( G \) by \( t(G) \).

Another way to think of thickness is decomposing the graph into the smallest possible number of planar graphs, each of which has the same vertex set, but all of which have disjoint edges sets chosen so that their union is the edge set of the original graph.

Proposition 6.2 A planar graph has thickness 1.

Proof: this is obvious, from the definitions.

Example 6.40 Prove that \( t(K_5) = 2 \).

Solution: We know that \( K_5 \) is not planar so its thickness is at least 2. The following drawing demonstrates it is exactly 2:

Notice that the two colors used to draw the edges are solid and dotted.

Proposition 6.3 \( t(K_n) \) is at most \( n - 3 \) for \( n \geq 4 \), otherwise it is 1.

Proof:

For \( n \in \{1, 2, 3, 4\} \) we know that \( K_n \) is planar and so the thickness is one. Notice that the drawing in Example 6.40 can be obtained by adding one vertex to a planar drawing of \( K_4 \) and connecting it to all other vertices with a new color of edge. This can always be done so the thickness of \( t(K_{n+1}) \leq t(K_n) + 1 \) and the proposition follows by induction. \( \square \)

In fact the thickness of the complete graph is lower than the upper bound given in Proposition 6.3, something we will explore in the exercises.

Proposition 6.4 Suppose that \( G \) is a graph and \( H \) is a subgraph of \( G \). Then

\[ t(H) \leq t(G) \]

Proof: this proof is left as an exercise.

Proposition 6.5 Suppose that \( G \) and \( H \) are graph so that \( G \) is the prism of \( H \). Then \( t(G) \leq t(H) + 1 \).

Proof:

Make two identical drawings of \( H \) that have edges colored so as to witness the thickness of \( H \). Using one additional color, put in the additional edges
needed to construct the prism. These can be drawn as straight line segments that do not intersect. This demonstrates that $t(G) \leq t(H) + 1$.

### 6.7.1 Embeddings in the Torus

Examine Figure 6.3. The first shape, labeled A, is a rectangle with its opposite sides labeled with two sorts of arrows. The arrows show which sides will be identified later. Shape B has the sides marked with the double arrows identified, to form a tube. In shape C the now circular sides marked with the single arrows have been identified to yield a surface that separates space into two pieces. This shape is called a torus. It is easiest, at first, to think of a torus as the surface of a donut. It is also called a surface of genus one, meaning it has one hole in it.

A key point implied by Figure 6.3 is that we can draw graphs on the surface of a torus by using a square with identified edges rather than going and finding modeling clay and a stylus. When drawing graphs on the surface of a torus using its representation as a rectangle it is important to remember that opposite edges are identified. This means these opposite edges are the same line segment, somehow drawn in two distinct locations.

**Example 6.41** Show that $K_5$ can be drawn on the surface of a torus without crossings.

**Solution:** Remembering to identify opposite sides of the rectangle as in Figure 6.3, here is a drawing of $K_5$ in the torus.
6.7. TOPOLOGICAL GRAPH THEORY

Definition 6.87 A non-crossing drawing of a graph in a surface is called an embedding of the graph in the surface.

Definition 6.88 A graph that can be embedded in the torus is called a toroidal graph. It is also called a graph of genus one.

Example 6.42 Show that $K_{3,3}$ is toroidal.

Solution:

\[ \]

Notice that, like non-crossing drawings of graphs in the plane, graphs that are witnesses that a graph is toroidal divide the torus into faces.

Definition 6.89 The toroidal dual of a graph is defined from the faces and edges of a non-crossing drawing in the torus in exact analogy to the way a planar dual is derived from a non-crossing drawing of a graph in the plane.

Now that we have a notion of an embedding of a graph in the torus, we may give an analog to Euler’s theorem.

Lemma 6.29 Suppose that we have an embedding of a graph in the torus with $v$ vertices, $e$ edges, and $f$ faces. Then

\[ v - e + f = 0 \]

Proof: this proof is left as an exercise.

The famous result on planar graphs is the 4-color theorem. There is an analogous result for the coloring of maps resulting from embeddings of graphs in the torus that we offer without proof.

Theorem 6.15 The 7-color Theorem A toroidal graph may be properly colored in seven or fewer colors.

As we will see in Problem 6.218, it is possible to draw seven countries on the surface of the torus that are all in mutual contact. We will now offer a direct proof that $K_5$ has a non-crossing drawing on the torus, designed primarily to help students develop intuition.

Example 6.43 The picture on the left shows a part of the surface of a sphere with what is called a handle attached to it. The handle is like an three-dimensional arch. It is not too much of a stretch to see that, by judiciously inflating, deflating, and stretching, that a sphere with a handle on it can be deformed into a torus.

The picture on the right is $K_5$ drawn with the smallest number of possible crossings: one. It is possible to place the handle so that one of the two edges that meet could pass over the other by using the handle as a bridge. This demonstrates that $K_5$ has an embedding in the torus.

Example 6.43 suggest the following consequence.

Lemma 6.30 All graphs have an embedding in a sphere with some number of handles attached.

Proof:

Make any drawing and add a handle as a bridge whenever two edges cross, \( \square \)

The proof of Lemma 6.30 is very unlikely to use an efficient number of handles. $K_7$ has at least nine crossings in any planar drawing but can be embedded in the surface of a sphere with one handle (see Problem 6.218). We now supply of couple of definitions that supply the context for doing such drawings efficiently.

Definition 6.90 A sphere with $k$ handles is called a surface of genus $k$. 


Definition 6.91 The genus of a graph $G$ is the smallest $k$ for which the graph can be embedded in
a surface of genus $k$.

Example 6.44 Below is an identification surface, constructed from an octagon, for a surface of genus
2 and a picture of a surface of genus 2.

To make a surface of genus two out of the octagon, the
arrows are matched up with corresponding numbers of
chevrons and with the arrows pointing in the same
direction.

One conclusion of the Robertson-Seymour theorem,
which we do not state here, is that there is a
finite list of graphs such that if $G$ is of genus $g$
then $G$ contains
a subgraph homeomorphic to graph on the list or,
alternatively, a subgraph with a minor isomorphic to
a graph on the list. For $g = 0$ this list is empty
and for $g = 1$ this list is known to be $K_{3,3}$ and $K_5$.
The list is not known for any $g \geq 2$ and is thought to
contain between thirty-and and perhaps hundreds
of graphs.

The analogs to the 4-color theorem are known. A cor-
rect conjecture of Heawood in 1890, proven by Ger-
hard Ringel and J. T. W. Youngs in 1968 says the follow-
ing.

Theorem 6.16 A map resulting from the embedding
of a graph $G$ into a surface of genus $g$ requires no
more than
$$\frac{7 + \sqrt{1 + 48g}}{2}$$
colors.

Problems

Problem 6.204 Prove that “$H$ is an edge contrac-
tion of $G$” is a partial order of the set of finite simple
diagrams.

Problem 6.205 Let $PD_{2n}$, be a $2n$-difference graph
with difference set $\{1, n\}$. This graph is called a poly-
gon with diagonals. Demonstrate for $n \geq 3$ that $PD_n$
is homeomorphic to $K_{3,3}$.

Problem 6.206 Prove that $H_n$ is homeomorphic to
$K_{3,3}$ iff $n \geq 4$.

Problem 6.207 Prove that every finite tree is home-
omorphic to a unique tree with no vertices of degree
two.

Problem 6.208 Prove that for any two complete
diagrams, one is a minor of the other.


Problem 6.210 Compute the toroidal dual of the
drawings of $K_5$ and $K_{3,3}$ given in Examples 6.41 and
6.42.

Problem 6.211 Verify that the crossing number of
$K_6$ is three.

Problem 6.212 Verify that the crossing number of
$K_7$ is nine.

Problem 6.213 Prove that the thickness of the Pe-
tersen graph is 2.

Problem 6.214 Find the thickness of $K_6$.

Problem 6.215 Find the thickness of $K_{4,4}$.

Problem 6.216 Prove Proposition 6.4.

Problem 6.217 Prove that $t(H_n) \leq n - 2$.

Problem 6.218 Find an embedding of $K_7$ in the
torus.

Problem 6.219 Find a toroidal map requiring ex-
actly 5 colors.

Problem 6.220 Find a toroidal map requiring ex-
actly 6 colors.

Problem 6.221 Find a toroidal map requiring ex-
actly 7 colors.

Problem 6.222 Construct an embedding of $K_8$ into
a surface of genus two.
6.8 Cages

An extremal graph is one that hits some boundary on the behavior of graphs. A complete graph, for example, has as many edges as it possibly can given its number of vertices. This section deals with a type of extremal graph called a cage.

Definition 6.92 An n-regular simple graph, $n \geq 2$, with girth $g$ that has the smallest possible number of vertices is an $(n,k)$-cage.

Example 6.45 Shown are examples of $(3,3)$-, $(3,4)$-, and $(3,5)$-cages.

![Cage Examples](image)

These cages are, in fact, unique. They are also already well known to us as $K_4$, $K_{3,3}$, and the Petersen graph. The fact that they are cages will be proven subsequently.

Lemma 6.31 A cage must be connected.

Proof: this proof is left as an exercise.

Lemma 6.32 The $(2,k)$-cage is $C_k$.

Proof:

To have girth $k$ the graph must have a subgraph that is a $k$-cycle. Since $C_k$ consists of a $k$-cycle it is as small as possible and so must be a $(2,k)$-cage. Since it is the only 2-regular graph on $k$ vertices it follows that it is also unique. □

Lemma 6.33 The $(n,3)$-cage is $K_{n+1}$.

Proof:

Note that $K_{n+1}$ has the required girth and degree. Since it is the smallest simple graph with its degree it is the cage. □

Lemma 6.34 The $(n,4)$-cage is $K_{n,n}$.

Proof:

First of all, note that since $K_{n,n}$ is bipartite, Theorem 6.1 tells us that all its cycles are of even length. It is rich in 4-cycles and cannot have any three cycles and so its girth is 4. Suppose the $G$ is is an $(n,4)$-cage and let $u, v \in V(G)$ and $\{u, v\} \in E(G)$. Since $G$ is $n$-regular it follows that $|\Gamma(u)| = |\Gamma(v)| = n$. This means that, in addition to one another, both $u$ and $v$ have $n - 1$ additional neighbors. If any of these neighbors were in common, a 3-cycle would result. We thus see that $|V(G)| \geq 2n$. Since $K_{n,n}$ has exactly $2n$ vertices it follows it is as few vertices as possible and we see it is an $(n,4)$-cage. □

A technique used in the proof of Lemma 6.34 generalizes to give a lower bound on the size of cages. The upper bound comes in two parts. The first part is for cages of odd girth.

Lemma 6.35 An $(n,2k+1)$-cage contains at least

$$1 + n \sum_{i=0}^{k-1} (n - 1)^i$$

vertices.

Proof:

Let $G$ be an $(n,2k+1)$-cage. Start at any vertex $v \in V(G)$. Divide the graph into levels based on their distance from $v$. There can be no cycle of length less than $k$ including $v$, so no two paths of length $k-1$ starting at $v$ can share any vertices; otherwise a cycle of length less than $k$ would form. Every vertex in $G$ has $n$ neighbors. This means that the first level contains $n$ vertices, the second contains $n(n-1)$, the third contains $n(n-1)^2$ and, in general, the $j$th level contains $n(n-1)^{j-1}$. The total number of levels before paths may converge is $k$ and so the sum given in the Lemma simply totals up the levels. □

Notice that the complete graph $K_{n+1}$, which is the $(n,3)$-cage, exactly hits this bound. What Lemma 6.35 does is to find a tree that must be a subgraph of the cage, given its degree and girth. Example 6.46 shows some of these trees.

Example 6.46 The various levels that the proof of Lemma 6.35 says must appear in a $(3,2k+1)$-cage for $K = 1, 2, 3$ are shown below. In a homework problem we saw that the center of a tree
Corollary 6.8 The Petersen graph is a (3, 5)-cage.

Proof:

For $2k + 1 = 5$, $k = 2$ and we see the lower bound given by Lemma 6.35 is 10, which the Petersen graph attains.

Lemma 6.36 An $(n, 2k)$-cage contains at least

$$2 \sum_{i=0}^{k-2} (n - 1)^i$$

vertices.

Proof:

The proof is similar to the proof of Lemma 6.35. Starting with the ends of an edge, rather than a single vertex, as level zero, build levels as before. All paths of length $k - 2$ starting at the two ends of the initial edge must be disjoint to avoid a $2k$-cycle, yielding $2(n - 1)^i$ vertices on levels 1 through $k - 2$. The sum given in the lemma is the total of the levels.

Notice that the size of $K_{n,n}$ is equal to this bound for the $(n, 4)$-cage, i.e. for $k = 2$.

Example 6.47 The various levels that the proof of Lemma 6.36 says must appear in a $(3, 2k)$-cage for $K = 2, 3, 4$ are shown below. Notice that these are edge centered trees.

Our next task is to construct a $(3, 6)$-cage. We will do so by modifying the Petersen graph.

Example 6.48 Verify that the two graphs below are isomorphic. In a homework problem we verified that the Petersen graph has 12 pentagons. In the left drawing below, notice that the vertices inserted into three of the edges of the Petersen graph cover one edge of all of these pentagons, enlarging them into hexagons. Verify, by applying Algorithm 6.1 that the new vertex adjacent to all three of these inserted vertices is not on any pentagons. Conclude that the resulting graph is of girth six. Note that 14 is the lower bound on the size of a six cage and so the graph below is a $(3, 6)$-cage.

This graph is called the Heawood graph and is, in fact, the unique $(3, 6)$-cage.

The Heawood graph can be constructed a number of different ways. We will first add some formality to the method used in Example 6.48 and then give some of the other presentations of the Heawood graph.

Definition 6.93 Suppose that $G$ is a graph and that $e_1, e_2 \in E(G)$. The edge insertion of $G$ at $e_1$ and $e_2$ is performed as follows. First perform a subdivision of both edges, inserting a new vertex of degree 2 into each. Now add a new edge joining the two new vertices. We denote the edge insertion by $G^+_{e_1,e_2}$.

Proposition 6.6 If $G$ is an edge insertion of $H$ and $H$ is 3-connected then so is $G$.

Proof:

Suppose that $G$ is not 3-connected, then there exist a vertex cut of size one or two. If this vertex
6.8. CAGES

cut does not involve the new vertices created by the edge insertion then it would have to be a vertex cut in $H$ and so has size 3 or more. Deleting both the new vertices has the same effect as removing two edges from $H$. Theorem 6.10 says that there are three vertex independent paths between any pair of vertices in $H$, including any two ends of the edges into which the new edge was inserted. Removing two edges interrupts at most two paths: $G$ remains connected if we delete the two new vertices. Similarly if we delete one of the new vertices and one other then the deletion of the one other vertex leaves a 2-connected subgraph of $H$ and the same argument demonstrates that deleting the one edge of $H$ associated with the new vertex that was delete leaves $G$ connected. The Proposition follows. □

**Definition 6.94** For a graph $G$ an edge deletion at an edge $e \in E(G)$ is done as follows. First the edge $e$ is removed. If either of its former ends are of degree 2 then that vertex $b$ is removed and the edges $\{a, b\}$ and $\{b, c\}$ incident on that vertex are replaced with a single edge $\{a, c\}$. This is called smoothing a vertex of degree 2. Edge deletion exactly reverses edge insertion when applied to an edge created by edge insertion. We denote the edge deletion by $G_e^{-}$. Review Example 6.48. It is possible to move from the Petersen graph to the Heawood graph with two edge insertions. The next example shows a number of connections between familiar graphs that may be made by edge insertion.

**Example 6.49** The diagrams below show a method of transforming $K_4$ into $K_{3,3}$ with a single edge insertion and $K_{3,3}$ into the Petersen graph with two edge insertions.

Notice that Example 6.48 and this example connect, via edge insertion, the (3, 3)- through (3, 6)- cages.

**Definition 6.95** If $G$ is a cubic graph and $v \in V(G)$ then vertex deletion at $v$ consists of removing $v$ and its incident edges from $G$ and then smoothing over the resulting degree two vertices.

**Lemma 6.37** The relation $H < G$ if $H$ is an edge deletion of $G$ partially orders the cubic graphs.

Proof: this proof is left as an exercise.

The next example give a geometric presentation of the Heawood graph. A third presentation appears as Problem 6.233.

**Example 6.50** Let $P = \{A, B, C, D, E, F, G\}$ and let $L = \{\{A, B, C\}, \{A, D, E\}, \{A, F, G\}, \{B, D, G\}, \{B, E, F\}, \{C, D, F\}, \{C, E, G\}\}$. This collection of sets has a number of remarkable properties. If we let $p$ be the set of points and $L$ be the set of lines then any two points uniquely determine a line and any two lines intersect in a unique point. This structure is an example of a finite projective plane called the Fano plane. Below is the Heawood graph, labeled with the points and lines of the Fano plane so that edges are between a line and a point on that line.
Notice that the points and line form a bipartition and so the Heawood graph is bipartite.

At this point we will stretch out a little into the linear algebra and geometry of finite fields in order to obtain an infinite collection of \((r, 6)\)-cages.

**Definition 6.96** For finite field \(F\) define a graph \(H(F)\) as follows. Let \(V = F^3\) be a three dimensional vector space over \(F\). Let \(A\) be the set of one-dimensional subspaces of \(V\) and let \(B\) be the set of two dimensional subspaces of \(V\).

\[
V(H(F)) = A \cup B
\]

and

\[
E(H(F)) = \{\{a, b\} : a \subset b\}
\]

**Lemma 6.38** The graph \(H(F)\) has \(2(q^2 + q + 1)\) vertices, with \(q = |F|\).

**Proof:**

There are \(q^3\) vectors in \(V = F^3\). Any two one-dimensional subspaces must contain \(q\) elements, but all these subspaces intersect in zero. There are, thus \(q - 1\) unique points in each such subspace. These points partition the \(q^3 - 1\) non-zero points. There are thus \(\frac{q^3 - 1}{q - 1} = q^2 + q + 1\) one-dimensional subspaces. Each two-dimensional subspace is orthogonal to a unique one-dimensional subspace and so the number of two dimensional subspaces is equal to the number of one-dimensional subspaces. The total vertices of \(H(F)\) is thus \(2(p^2 + p + 1)\). □

**Lemma 6.39** The graph \(H(F)\) is bipartite of girth six.

**Proof:**

Notice that the sets \(A\) and \(B\) from Definition 6.96 form a bipartition. Recall that two vectors \(\vec{u}\) and \(\vec{v}\) not in the same one-subspace generate a two-subspace that itself contains the one-subspaces generates by \(\vec{u}\) and \(\vec{v}\) respectively. Thus any two members of \(A\) are mutually adjacent to a single element of \(B\). This makes it impossible for \(H(F)\) to contain a 4-cycle. If \(a, b, c \in A\) and \(A = \langle a, b \rangle, B = \langle a, c \rangle, C = \langle b, c \rangle \in B\) then \(aAbBcC\) form a 6-cycle in \(H(F)\). □

**Theorem 6.17** \(H(F)\) is a \((q + 1, 6)\)-cage.

**Proof:**

Adopt the terms of Definition 6.96. There are \(q^2 - 1\) non-zero vectors in any member of \(B\). They can partitioned into members of \(A\) with zero removed each of which contain \(q - 1\) vectors. This means the number of \(1\)-subspaces of a \(2\)-subspace of \(F^3\), and hence the degree of each element of \(B\), is \(q - 1\). On the other hand, there are \(q^3 - q\) non-zero elements in \(V\) and each one-subspace contains \(q - 1\) of these. This means that there are \(q^3 - q\) nonzero elements outside of a given one-space. Any of these, combined with a one space, generates a two-space. A two space containing a one-space has, similarly, \((q^2 - 1) - (q - 1) = q^2 - q\) nonzero elements outside of the one-space. This means that number of two-spaces on a given line is \(\frac{q^2 - q}{q - 1} = q + 1\) and so we see that \(H(F)\) is \(q + 1\)-regular. We now apply the formula for finite geometric series to Lemma 6.36 when \(2k = 6\) and see that the minimum number of vertices in a \((q + 1, 6)\) cage is \(2(q^2 + q + 1)\) which finishes the proof. □

Recall from Chapter 5 that there is one finite field \(F\) of order \(q\) when \(q\) is a positive power of a prime number and no finite fields otherwise. This means that an infinite number of \((n, 6)\)-cages are known and have a size that realized the lower bound: those for which \(n - 1\) is a prime power. Notice that the Heawood graph is \(H(F_2)\).

We now turn again to \(3\)-cages or \textit{cubic cages}. The first cubic cage that does not hit the lower bound is the \((3, 7)\)-cage. This cage is also unique but is also the first that is not vertex transitive. It has 24 vertices when the lower bound is 22.

**Example 6.51** The \((3, 7)\)-cage, shown below, is the McGee graph.
There is a simple method for creating the McGee graph from the generalized Petersen graph $P_{5,3}$ by edge insertion.

**Lemma 6.40** View $P_{5,3}$ as being made of an inner and outer 8-cycle joined by spokes. The spokes come in opposite pairs. If we insert an edge in pairs of opposite spokes, as shown above, then we obtain the McGee graph.

Proof: This proof is left as an exercise.

Somewhat surprisingly, the (3,8)-cage both hits the lower bound (of 30) and is vertex transitive. It is shown below and is called the Tutte-Coxeter or Levi graph. We will give three constructions.

**Example 6.52** Below is the unique (3,8)-cage.

The Tutte-Coxeter or Levi graph.

**Lemma 6.41** In Example 6.48 a careful pair of edge insertions into three edges transformed the Petersen graph into the Heawood graph. The set of edges used appears five times under an obvious five-fold rotational symmetry of the Petersen graph. Performing the Heawood-construction insertion simultaneously into all five sets of edges yields the Tutte-Coxeter graph.

Proof: This proof is left as an exercise.

**Lemma 6.42** This lemma is similar to Lemma 6.40. The Tutte-Coxeter graph may be obtained by simultaneous edge insertion into opposite pairs of spokes of the generalized Petersen graph $P_{10,3}$.

Proof: This proof is left as an exercise.

**Example 6.53** If we have six points \{a, b, c, d, e, f\} then there are \(\binom{6}{3} = 15\) pairs of points. There are also 15 ways to divide six points into three distinct pairs:

| (a b)(c d)(e f) | (a b)(c e)(d f) | (a b)(c f)(d e) |
| (a c)(b d)(e f) | (a c)(b e)(d f) | (a c)(b f)(d e) |
| (a d)(b e)(c f) | (a d)(b f)(c e) | (a d)(b c)(e f) |
| (a e)(b c)(d f) | (a e)(b d)(c f) | (a e)(b f)(c d) |
| (a f)(b c)(d e) | (a f)(b d)(c e) | (a f)(b e)(c d) |
Construct a graph $G$ in which $V(G)$ consists of all the pairs and triples of distinct pairs above. Edges go from a pair to a triple of pairs if the pair is a member of the triple. This is constructively a cubic, bipartite graph on 30 vertices. The bipartition is into pairs and triples of pairs. If we can demonstrate it is of girth eight then it must be a $(3, 8)$-cage by virtue of Lemma 6.36.

Since $G$ is bipartite every other vertex of any cycle is a pair and cycles must be of even length. Notice that knowing to pairs in a triple dictates the third. A 4-cycle would thus need to use the same triple twice and thus does not exist. Suppose that there is a 6-cycle in $G$. Then it contains three pairs. If these pairs are disjoint then they are all joined to their union and we see there are 4-cycles, something we have already managed to dismiss. This means that two of the pairs must intersect: but this is also impossible because they are both adjacent to some one triple which is made of three disjoint pairs. It follows there are no 6-cycles. Since the graph is bipartite there are no odd-cycles and we see that it has girth 8 or more. A witness that there are 8-cycles is:

$$(ab) - (ab)(cd)(ef) = (ef) - (ac)(bd)(ef) = (bd) -$$

$$(af)(bd)(ce) = (ce) - (ab)(ce)(df) = (ab)$$

And so we see $G$ is a $(3, 8)$-cage. That is it unique we offer without proof.

**Proposition 6.7** The Tutte-Coxeter graph is vertex transitive.

**Proof:**

Let $G$ be the Tutte-Coxeter graph as was constructed in Example 6.53. Notice that any permutation of the set $\{a, b, c, d, e, f\}$, in effect, renaming the vertices of $G$ in a manner that preserves the adjacency relationship. Call these renaming automorphisms. Notice that the permutation $(x a y b)(z w)$ exchanges the pairs $(x y)$ and $(a b)$ for any $x$ and $y$ and so renaming automorphisms are transitive on the vertices that are pairs of points. View the vertices that are triples as being triple-two cycles in $S_6$. The proposition 4.22 says that they are conjugate and we recall that conjugation acts by renaming. This means that renaming automorphisms are also transitive on the vertices that are triples. We have now shown there are at most two orbits of $Aut(G)$ on $G$: one encompassing the pairs and the other the triples of pairs. Examine the drawing of $G$ in Example 6.53. There is a left-right symmetry that exchanges the ends of several edges; this automorphism exchanges pairs with triples of pairs. This shows $Aut(G)$ has a single orbit and the proposition follows.

We now summarize the remainder of what is known about the $(3, k)$ cages. There are 18 different $(3, 9)$-cages with 58 vertices. There are 3 different $(3, 10)$-cages with 70 vertices. There is a unique $(3, 11)$-cage with 112 vertices. There is a unique $(3, 12)$-cage with 126 vertices, the last that hits the lower bound and the last where the number of vertices is known.

**6.8.1 Distance Separating Maps**

**Definition 6.97** Suppose that $M$ is a metric space with distance function $d$. Then a $k$-separating map on $M$ is a bijection $f : M \rightarrow M$ with the property that

$$d(x, y) + d(f(x), f(y)) \geq k$$

$\forall x \neq y \in M$.

If $M$ is the vertex set of a graph under the path metric then any bijection of the vertex set is trivially 2-separating, since the minimum distance between pairs of points is 1. If a $k$-separating map on a graph has $k \geq 3$ we call it a distance separating map.

**Proposition 6.8** Consider the metric space $\mathbb{R}^n$ under the Euclidean metric. Then for all $k > 0$ there is no $k$-separating map.

**Proof:**

Let $B$ be a ball of diameter $k/2$. Then if $f$ is a $k$-separating map each point in this ball must end up a distance of $k/2$ from each other point in the ball. There are uncountably many points in the ball. Since each pair of points is separated by a positive distance, it follows that each is in a unique ball of positive radius. This means that there are an uncountable set of balls of positive radius. Notice, however, that every sphere of positive radius contains a point all of whose coordinates are rational. Since $\mathbb{Q}^n$ is a countable set,
this is impossible. The proposition follows from this contradiction. □

**Definition 6.98** Let \( f \) be a bijection of the vertex set of a graph \( G \) with itself. Then the \( f \)-prism of \( G \) is like a standard prism of \( G \) but the edges between the copies of \( G \) follow \( f \).

If \( f : V(G) \rightarrow V(G) \) is the identity map then the \( f \)-prism of \( G \) is the standard prism. More interesting things happen when \( f \) is a distance separating map.

**Lemma 6.43** Suppose that \( G \) is a graph and that \( f : V(G) \rightarrow V(G) \) takes adjacent pairs of vertices to non-adjacent pairs of vertices. Then \( f \) is 3-separating.

**Proof:**

Let \( x \neq y \in V(G) \) and let \( d \) be the path metric. The only way for \( d(x,y) + d(f(x),f(y)) \) is if both of the summands are one. The property of mapping adjacent pairs to non-adjacent pairs tells us that one of the summands is always at least two. □

**Example 6.54** All arithmetic in this example is done \((\bmod 5)\). Let \( C_5 \) be constructed with \( V(C_5) = \mathbb{Z}_5 \) so that adjacent vertices differ by \( \pm 1(\bmod 5) \) and let \( f : \mathbb{Z}_5 \rightarrow \mathbb{Z}_5 \) be given by \( f(x) = 2x \). Then \( f \) is a 3-separating map and the \( f \)-prism of \( \mathbb{Z}_5 \) is the Petersen graph. First we note that \( \text{GCD}(5,2) = 1 \) tells us that multiplication by 2 is a bijection of \( \mathbb{Z}_5 \). Notice that \( f \) takes \((x,x+1)\) to \((2x,2x+2)\) and so Lemma 6.43 tells us \( f \) is 3-separating. To see that the prism is the Petersen graph, examine the following drawing.

![Petersen Graph Diagram](image)

**Lemma 6.44** Suppose that \( G \) is a graph of girth 5 and that \( f \) is a 3-separating map on \( V(G) \). Then the \( f \)-prism of \( G \) has girth 5 as well.

**Proof:**

Since \( G \) is of girth 5 there are no 3 or 4-cycles that do not involve the edges that run between the copies of \( G \) in the prism. To complete a cycle at least two such edges must be involved. Vertices that are adjacent in one copy of \( G \) are mapped to vertices that are not adjacent in another. This means that the shortest cycle involving two between-copy edges is a 5-cycle and the lemma follows. □

Notice that Example 6.54 illustrates the method of producing graphs of girth five implied by Lemma 6.44. This technique may be continued to higher girths for a time but runs aground at girth 8. Problem 6.43 forces the student to demonstrate that generalized Petersen graphs have a maximum girth of 8.

**Theorem 6.18** Suppose that \( G \) is a graph of girth \( k \leq 8 \) and that \( f \) is a \( k-2 \)-separating map. Then the \( f \)-prism of \( G \) has girth \( k \).

**Proof:**

There are no cycles of length less than \( k \) in either copy of \( G \) and so any such cycle must involve edges that run between copies of the graphs. The need to make a cycle means that the number of such cross-edges must be even. If such a cycle involves two such edges then the sum of the length of the shortest paths joining the ends of these two edges in each copy of \( G \) is at least \( k-2 \) because \( f \) is \( k-2 \)-separating. If such a cycle involved four cross-edges its minimum length would be eight because at least every other edge of the cycle must be a non-cross edge. Cycle involving more than 4 cross edges must similarly have length exceeding 8. □

**Problems**

**Problem 6.223** Prove Lemma 6.31.

**Problem 6.224** Tabulate the lower bounds on \((3,k)\)-cages by Lemmas 6.35 and 6.36.

**Problem 6.225** Find a copy of the tree for \( k = 2 \) from Example 6.46 in the Petersen graph. What structure do the remaining edges form?
Problem 6.226 Find a copy of the tree for \( k = 3 \) from Example 6.47 in the Heawood graph. What structure do the remaining edges form?

Problem 6.227 Prove that all single edge deletions of the Petersen graph are isomorphic to the octagon with diagonals.

Problem 6.228 Suppose \( H \) is the Heawood graph. Show that any vertex deletion of the Heawood graph is the Petersen graph.

Problem 6.229 Prove that he \( (3, k) \)-cages are 3-connected for \( k = 3, 4, 5, 6 \), and 8. Hint: do not do it directly.

Problem 6.230 Find six edge insertions that transform \( H_5 \) into the dodecahedron. Hint: it can be done by inserting a new vertex into every edge once.


Problem 6.232 Find an embedding of the Petersen graph in the torus so that \( K_5 \) (possibly with multiple edges) is the topological dual.

Problem 6.233 Find an embedding of \( K_7 \) in the torus so that the topological dual is the Heawood graph.

Problem 6.234 Prove that the Heawood graph is vertex transitive. Hint: you have several presentations of the graph available. Examine them to see if any are helpful.

Problem 6.235 Take a drawing of the McGee graph and mark the vertices and edges of the tree that is used to prove that the lower bound on the size of a \( (3, 7) \)-cage is 22 vertices. If you delete all the vertices of the McGee graph that are non-leaves of this tree what is the remaining structure that you obtain?


Problem 6.238 Prove Lemma 6.42.

Problem 6.239 Prove that the inverse of a \( k \)-separating map is also \( k \)-separating.

Problem 6.240 Suppose \( 3 \leq n \leq 15 \). Prove that no 6-separating map exists on \( C_n \).