University of Guelph
Mathematics Post-Secondary Preparation Package
(MP³)

SOLUTIONS

(i) \cot^2(\theta) + 1 =

(ii) \tan(-\theta) =

(iii) \tan\left(\frac{\pi}{2} - \theta\right) =

(iv) \tan(A \pm B) =

(v) \tan(2A) =

\[ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]
Solutions Part I

Brushing up on

Numerical Skills
1) Adding and Subtracting Fractions

**Problem:** Evaluate without a calculator!

(i) \( \frac{2}{3} + \frac{5}{6} \)  
(ii) \( \frac{2}{5} - \frac{3}{4} \)  
(iii) \( \frac{1}{6a} + \frac{2}{3a} - \frac{5}{12a} \)

**Solution:**

(i) \( \frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{17}{6} = \frac{21}{6} = \frac{7}{2} = 3 \frac{1}{2} \)

(ii) \( \frac{2}{5} - \frac{3}{4} = \frac{68}{20} - \frac{55}{20} = \frac{13}{20} \)  

(iii) \( \frac{1}{6a} + \frac{2}{3a} - \frac{5}{12a} = \frac{2}{12a} + \frac{8}{12a} - \frac{5}{12a} = \frac{5}{12a} \)

**Notes:** Use the “lowest” common denominator.

**Common error:** Not using the lowest common denominator leads to larger numerators and more possibility of making mechanical errors.

**Practice:** (i) \( \frac{3}{4} + 1\frac{1}{2} \)  
(ii) \( \frac{1}{7} - 1\frac{2}{3} \)  
(iii) \( \frac{3}{5} + \frac{2}{4} - \frac{1}{30} \)

**Answer:** (i) \( \frac{9}{4} = 2\frac{1}{4} \)  
(ii) \( \frac{31}{21} = 1\frac{10}{21} \)  
(iii) \( \frac{16}{15} \)

**A Great Website for More Detail:** [http://www.math.com/](http://www.math.com/)
2) Multiplying and Dividing Fractions

Problem: Evaluate without a calculator!

(i) \( \frac{5}{3} \times \frac{12}{25} \)  
(ii) \( \frac{2}{5} \times \frac{3}{4} \)  
(iii) \( 3 \times \frac{4}{5} \)  
(iv) \( \frac{5}{3} \times \frac{25}{12} \)  
(v) \( \frac{4}{3} \times \frac{3}{5} \)  
(vi) \( \frac{4}{3} \times \frac{3}{5} \)

Solution:

(i) \( \frac{5}{3} \times \frac{4}{12} \times \frac{25}{75} = \frac{1 \times 4}{1 \times 5} = \frac{4}{5} \)  
(ii) \( \frac{2}{5} \times \frac{3}{4} = \frac{7 \times 3}{20} = \frac{21}{20} = \frac{1 \times 1}{20} \)

(iii) \( 3 \times \frac{4}{5} \times \frac{3}{1} = \frac{12}{5} = 2 \frac{2}{5} \)  
(iv) \( \frac{5}{3} \times \frac{25}{12} \)  
(v) \( \frac{4}{3} \times \frac{3}{5} \)  
(vi) \( \frac{4}{3} \times \frac{3}{5} \)

Note: Common denominators are irrelevant when multiplying and dividing.

Common error: \( \frac{a}{b} \times \frac{c}{1} = \frac{a \times c}{b} \)

Practice: (i) \( \frac{3}{4} \times \frac{8}{15} \)  
(ii) \( \frac{2}{7} \times \frac{12}{21} \)  
(iii) \( \frac{22}{3} \times \frac{11}{3} \)  
(iv) \( \frac{22}{3} \times \frac{11}{11} \)

Answer: (i) \( \frac{2}{5} \)  
(ii) \( \frac{1}{2} \)  
(iii) \( \frac{2}{3} \)  
(iv) \( \frac{242}{3} \)

A Great Website for More Detail: http://www.math.com
3) Working with Decimals.

**Problem:** Evaluate without a calculator!

(i) $1.02 + .023$  
(ii) $1.02 - 2.57$  
(iii) $1.2 \times .05$  
(iv) $\frac{4.291}{3}$

(v) $\frac{4.291}{0.3}$  
(vi) $\frac{0.004291}{0.03}$

**Solution:**

(i) $1.02 + .023 = 1.043$

(ii) $1.02 - 2.57 = -(2.57 - 1.02) = -1.55$

(iii) $1.2 \times .05 = 0.06$

(iv) $\frac{4.291}{3} = 1.430\overline{3} = 1.4303333\ldots$

(v) $\frac{4.291}{0.3} = \frac{4.291 \times 10}{0.3 \times 10} = \frac{42.91}{3} = 14.30\overline{3}$

(vi) $\frac{0.004291}{0.03} = \frac{0.004291 \times 100}{0.03 \times 100} = \frac{.4291}{3} = 0.1430\overline{3}$

**Note:** When dividing by a decimal, most of us were taught to move the decimal the same number of places to the right in questions such as (iv) and (v). What we are really doing is multiplying the numerator and denominator by the same power of 10.

**Common error:** Running to our calculators when there is a decimal in the denominator because division by a decimal is scary.

**Practice:** (i) $21.021 - 25.555$  
(ii) $30 \times 0.00005$  
(iii) $\frac{3.6}{.000018}$

**Answer:** (i) $-4.534$  
(ii) $0.0015$  
(iii) $200000$

**A Great Website for More Detail:** [http://www.math.com](http://www.math.com)
4) Roots and Radicals

Problems: 1) Write as simplified mixed radicals:
   (i) $\sqrt{40}$  (ii) $2\sqrt{27}$  (iii) $\sqrt{x^4y^7}$  (iv) $\sqrt[3]{x^4y^7}$

2) Write as entire radicals: (i) $3\sqrt{2}$  (ii) $\frac{4}{9}$  (iii) $xy^4\sqrt{xy}$  (iv) $xy^4\sqrt[4]{xy}$

3) Evaluate: (i) $\sqrt[3]{121}$  (ii) $\left(\frac{27}{64}\right)^{\frac{2}{3}}$  (iii) $32^{1/5}$  (iv) $(-32)^{1/5}$  (v) $(-64)^{1/6}$

Solution:
1) (i) $2\sqrt{10}$  (ii) $6\sqrt{3}$  (iii) $x^2y^3\sqrt{y}$  (iv) $\frac{1}{3\sqrt{x^4y^7}} = xy^2\sqrt[3]{xy}$

2) (i) $\sqrt{18}$  (ii) $\frac{16}{\sqrt{81}}$  (iii) $\sqrt[3]{x^3y^9}$  (iv) $\sqrt[3]{x^4y^{13}}$

3) (i) $11$  (ii) $\frac{9}{16}$  (iii) $2$  (iv) $-2$  (v) does not exist (as a “real number”)

Note: Radical signs are just special cases of the more common exponents signs. For example, $\sqrt[3]{x} = x^{\frac{1}{3}}$.

Common error: In real numbers, we can not have the square root or fourth root or sixth root, etc., of a negative number. We CAN have the third root or the fifth root or the seventh root, etc., of a negative number. By the way, many calculators give an error message when ask for the odd root of a negative. Boo to the calculator’s programmer!

Practice: 1) Write as mixed radicals: (i) $\sqrt{1000}$  (ii) $2\sqrt{27}$  (iii) $\sqrt{x^{12}y^9}$  (iv) $\sqrt[5]{x^{15}y^7}$

2) Write as entire radicals: (i) $5\sqrt{5}$  (ii) $\frac{4}{9}$  (iii) $xy^5\sqrt{x^{1/3}y}$  (iv) $x^2y^4\sqrt[4]{x}$

3) Evaluate: (i) $\sqrt[3]{-27}$  (ii) $\left(\frac{36}{25}\right)^{\frac{3}{2}}$  (iii) $(-243)^{1/5}$  (iv) $(-81)^{3/4}$

Answers: 1) (i) $10\sqrt{10}$  (ii) $6\sqrt{5}$  (iii) $x^6y^4\sqrt[4]{y}$  (iv) $x^3y\sqrt[5]{y^2}$

2) (i) $\sqrt[3]{125}$  (ii) $\frac{16}{\sqrt{81}}$  (iii) $\sqrt[3]{x^{7/3}y^{11}}$  (iv) $\sqrt[4]{x^9y^{16}}$

3) (i) $-3$  (ii) $\frac{216}{125}$  (iii) $-3$  (iv) does not exist

A Great Website for More Detail: Gov.pe.ca/educ/docs/curriculum/521Bunit1.pdf
5) Absolute Value

Problems: 1) Evaluate (i) $|10|$  (ii) $|-10|$  (iii) $|0|

2) Write $|x|$ without absolute value bars if (i) $x > 0$  (ii) $x < 0$

3) Draw the graph of $y = |x|$

Solution: 1)(i) 10  (ii) 10  (iii) 0  2)(i) $|x| = x$  (ii) $|x| = -x$

3) ![Graph of $y = |x|$]

Note: When, for $x < 0$, we write $|x| = -x$, remember there is a negative inside the $x$!

Common error: Assuming $|x| = x$. This may or may not be true depending on $x$.
For example, if $x = 3$, then $|x| = -3 = 3$. But if $x = -3$, then $|x| = -(-3) = 3 \neq x$; in fact, here $|x| = -x$. It depends on $x$.

Practice: 1) Evaluate (i) $|-.001|$  (ii) $|.001|$  (iii) $|-0|

2) Write $y = |x - 1|$ without using absolute value notation and graph the function.

Answers: 1)(i) .001  (ii) .001  (iii) 0

2) $y = \begin{cases} -(x-1), & \text{if } x < 1 \\ x - 1, & \text{if } x \geq 1 \end{cases}$

A Great Website for More Detail: En.wikipedia.org/wiki/Absolute_value

MP$^3$ Section I: Brushing up on Numerical Skills 6
Solutions Part II

Lines

and

Slopes
1) Finding the Slope of a Line

**Problem:** Find the slope of the line joining (2,5) to (7,4).

**Solution:**

\[
\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 5}{7 - 2} = -\frac{1}{5}
\]

**Note:** It doesn’t matter which point you use as \((x_1, y_1)\).

**Common error:**

\[
\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_1 - x_2} = \frac{4 - 5}{2 - 7} = \frac{1}{5}
\]

**Practice:** Find the slope of the line joining \((-2,8)\) to \((4,-5)\).

**Answer:** slope = \(-\frac{13}{6}\)

**A Great Website for More Detail:** [purplemath.com/modules/strtlneq.htm](http://purplemath.com/modules/strtlneq.htm)
2) Parallel and Perpendicular Slopes

Problem: Find the slope of the line

(i) parallel (ii) perpendicular
to a line $l$ with slope $2$.

Solution: (i) Parallel slope = $2$  (ii) Perpendicular slope $= -\frac{1}{2}$.

Note: A line with slope 0 is horizontal. In this case, the perpendicular line, which is vertical, has undefined (or infinite) slope. Apart from the case of horizontal/vertical lines, the slope of a line perpendicular to a line with slope $m$ satisfies $m_{\text{perpendicular}} = -\frac{1}{m}$, that is, these slopes are “negative reciprocals”.

Common error: Perpendicular slope $= \frac{1}{2}$.

Practice: Find the slope of a line (i) parallel (ii) perpendicular to a line with slope $-\frac{3}{2}$.

Answer: (i) $-\frac{3}{2}$ (ii) $\frac{2}{3}$

A Great Website for More Detail: purplemath.com/modules/strtlneq.htm
3) Interpreting Slope

Problem:
(i) Match the slopes $0, \frac{1}{2}, 1, 2$ with the lines $l_1, l_2, l_3, l_4$.
(ii) Match the slopes $-\frac{1}{3}, -1, -3$ with the lines $l_5, l_6, l_7$.

Solution:
(i) $\begin{array}{cccc}
\text{line} & l_1 & l_2 & l_3 & l_4 \\
\text{slope} & 1 & \frac{1}{2} & 2 & 0 \\
\end{array}$
(ii) $\begin{array}{cccc}
\text{line} & l_5 & l_6 & l_7 \\
\text{slope} & -1 & -\frac{1}{3} & -3 \\
\end{array}$

Notes: For positive slope, as $x$ increases, $y$ increases. For negative slope, as $x$ increases, $y$ decreases. As you walk to the right, with positive slope you are walking uphill. With negative slope, you are walking downhill.

Common error: Misidentifying slope because the axes are not scaled one to one.

Practice: Match the slopes 0 and undefined (or $\infty$) with the lines $l_1$ and $l_2$.

Answer: $l_1$ has undefined (or infinite) slope and $l_2$ has 0 slope.

A Great Website for More Detail: algebra-online.com/slope-lines-1.htm
4) Finding the Slope and Intercepts From the Equation of a Line

**Problem:** Find the slope and \( x \) and \( y \) intercepts for each of the following lines:

(i) \( y = -2x + 5 \)  
(ii) \( 6x + 2y = 1 \)  
(iii) \( y = 5 \)  
(iv) \( x = 1 \)

**Solution:** In the equation \( y = mx + b \), \( m \) is the slope and \( b \) is the \( y \) intercept.

(i) \( y = -2x + 5 \): \( m = -2 \) and \( b = 5 \). To find the \( x \) intercept, set \(-2x + 5 = 0 \Rightarrow x = \frac{5}{2}\).

(ii) \( 6x + 2y = 1 \): \( x \) intercept \( \Rightarrow 6x = 1 \Rightarrow x = \frac{1}{6} \)  
\( y \) intercept \( \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2} \)  
For solving slope \( \Rightarrow y = -3x + \frac{1}{2} \Rightarrow m = -3 \)

(iii) \( y = 5 = 0x + 5 \)  
This is a horizontal line, that is, a line parallel to the \( x \) axis. The slope \( m \) is 0, the \( y \) intercept is 5 and there is no \( x \) intercept.

(iv) \( x = 1 \)  
This is a vertical line, that is, a line parallel to the \( y \) axis. The slope is infinite (or undefined). The \( x \) intercept is 1 and there is no \( y \) intercept.

**Note:** The \( y \) intercept corresponds to the point \((0, y)\) on the line and the \( x \) intercept corresponds to \((x,0)\).

**Common error:** Confusing the \( y \) intercept \( b \) with the point \((0,b)\).

**Practice:** Find the slope and \( x \) and \( y \) intercepts of the line \( y = -2x + 6 \).

**Answer:** \( m = -2; \ x \) intercept = 3; \( y \) intercept = 6

**A Great Website for More Detail:**  
Math.com/school/subject2/lessons/S2U4L2GL.html

MP3 Section II: Lines and Slopes 5
5) Finding the Equation of a Line Given Two Points

Problem: Find the equation of the line joining (4,5) to (7,14). Draw the graph.

Solution: The easiest formula for finding the equation of a line with slope \( m \) and point \( (x_1, y_1) \) is \( y - y_1 = m(x - x_1) \).

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14 - 5}{7 - 4} = 3 \quad \text{Use (4,5) as } (x_1, y_1).\]

\[
\therefore y - 5 = 3(x - 4) \text{ and so } y = 3x - 7
\]

Notes: It doesn’t matter which point you use for \( (x_1, y_1) \). If you are given the slope, you only need one point.

Common error: \( y - 5 = 3(x - 4) \) and so \( y = 3x - 17 \)

Practice: Find the equation of the line joining \((-2,1)\) to \((4,-5)\). Draw the graph.

Answer: \( y = -x - 1 \)

A Great Website for More Detail: Math.com/school/subject2/lessons/S2U4L2DP.html
6) Finding the Equation of a Line Given the Slope and a Point

**Problem:** Find the equation of the line with slope \( m = -2 \) passing through the point \((-4, 5)\). Draw the graph.

**Solution:** The easiest formula for finding the equation of a line with slope \( m \) and point \((x_i, y_i)\) is \( y - y_i = m(x - x_i) \).

\[
m = -2 \text{ and } (x_i, y_i) = (-4, 5)
\]

\[
\therefore y - 5 = -2(x + 4) \text{ and so } y = -2x - 3.
\]

**Notes:** Sometimes, you are given a “disguised” point. For example, if we were given \( x \) intercept \( \frac{13}{2} \), we would use the point \( \left(\frac{13}{2}, 0\right) \).

**Common error:** \( y - 5 = -2(x - 4) \) and so \( y = 3x + 3 \)

**Practice:** Find the equation of the line with slope \( m = 5 \) and with \( x \) intercept \(-4\). Draw the graph.

**Answer:** \( y = 5x + 20 \)

**A Great Website for More Detail:** [Purplemath.com/modules/strtlneq.htm](http://Purplemath.com/modules/strtlneq.htm)
7) Graphing Linear Inequalities

**Problem:** Show by shading the region in the \(xy\) plane that satisfies
(i) \(x + y \leq 3\)   
(ii) \(2x - y > 4\).

**Solution:** (i) Draw \(x + y = 3\).  
Test (0,0) in this equation  
Left Side = 0; Right Side = 3;  
0 \(\leq\) 3 is true. So the shaded region is on the same side of the line as (0,0).

(ii) Draw \(2x - y = 4\).  
Test (0,0) in this equation.  
Left Side = 0; Right Side = 4;  
0 > 4 is false. So the shaded region is on the opposite side of the line as (0,0).

**Note:** For linear inequalities, draw the line first, dotted if the sign is < or >, solid if the sign is \(\leq\) or \(\geq\). Then test a point on one side of the line. If the point makes the inequality true, shade that portion of the plane. If not, shade the portion on the other side of the line.

**Common error:** Testing a point which is on the line. This won’t answer the inequality question. You won’t get LS < RS or LS > RS; you will get LS = RS, which is no help!

**Practice:** Show by shading the region in the \(xy\) plane that satisfies \(x + 2y > 1\).

**Answer:**

**A Great Website for More Detail:** [Purplemath.com/modules/syslneq.htm](http://Purplemath.com/modules/syslneq.htm)
Solutions Part III

Algebraic

\[ ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ |x| < a, \ a > 0 \iff -a < x < a \]

\[ (a + b)^2 = a^2 + 2ab + b^2 \]

\[ (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \]

Skills

\[
\begin{align*}
\frac{x^2 - 5x + 6}{x+1} & \quad x^3 - 4x^2 + x + 6 \\
\frac{-(x^3 + x^2)}{} & \quad \therefore, \quad x^3 - 4x^2 + x + 6 = (x+1)(x^2 - 5x + 6) \\
\frac{-5x^2 + x + 6}{-(-5x^2 - 5x)} & \\
\frac{-(-5x^2 - 5x)}{} & \\
\frac{6x + 6}{-(6x + 6)} & \\
\frac{6x + 6}{0} &
\end{align*}
\]

MP³ Section III: Algebraic Skills 1
1) Adding and Subtracting Like Terms

Problem: Simplify: (i) $4st^2 + 2t^3s$

(ii) $(x^2 - 3xy + 7x - 1) + (2x^2 - xy - 3y - 4)$

(iii) $3(x + z) + 7(x + z) - y(x + z)$

Solution:

(i) $6st^2$

(ii) $(x^2 - 3xy + 7x - 1) + (2x^2 - xy - 3y - 4) = 3x^2 - 4xy + 7x - 3y - 5$

(iii) $3(x + z) + 7(x + z) - y(x + z) = (10 - y)(x + z)$

Notes: Terms are factors “glued” together with multiplication and division and are separated by addition and subtraction. In (iii), $(x + z)$ is a factor of each term. Don’t be confused by the fact that it is composed of two terms inside the brackets.

Common error: In (i),

$4st^2 + 2t^3s$ are not like terms and so we cannot simplify the expression.

Practice: Simplify: $(4w + 3wx - 2) - (2w - 3wx + 1)$

Answer: $2w + 6wx - 3$

A Great Website for More Detail: cstl.syr.edu/fipse/Algebra/Unit2/plusminu.htm
2) Multiplying Binomials

Problem: Expand: (i) \((3x+1)(2x-5)\)  (ii) \((2a+3b)^2\)

Solution:

(i) \((3x+1)(2x-5) = 6x^2 - 15x + 2x - 5 = 6x^2 - 13x - 5\)
(ii) \((2a+3b)^2 = (2a + 3b)(2a + 3b) = 4a^2 + 12ab + 9b^2\)

Notes: Each separate term in the first bracket is multiplied with each term in the second. Watch the signs!

Common error: In (ii), \((2a + 3b)^2 = 4a^2 + 9b^2\)

Practice: Expand: (i) \((a - 2b)(a + 2b)\)  (ii) \((5w - 2)^2\)

Answer: (i) \(a^2 - 4b^2\) (‘difference of squares’)  (ii) \(25w^2 - 20w + 4\)

A Great Website for More Detail:

regentsprep.org/regents/math/polymult/Smul_bin.htm
3) Multiplying Binomials and Trinomials

**Problem:** Expand: (i) \((x^2 + 3x + 1)(2x - 5)\)  (ii) \((a + b + c)^2\)

**Solution:**
(i) \((x^2 + 3x + 1)(2x - 5) = 2x^3 - 5x^2 + 6x^2 - 15x + 2x - 5 = 2x^3 + x^2 - 13x - 5\)
(ii) \((a + b + c)^2 = (a + b + c)(a + b + c) = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\)

**Notes:** The number of terms when you expand is the product of the number of terms in each bracket, that is, until you simplify. So there are \((3)(2) = 6\) terms in (i) and \((3)(3) = 9\) terms in (ii) before simplification.

**Common error:** In (ii), \((a + b + c)^2 = a^2 + b^2 + c^2\)

**Practice:** Expand: (i) \((a + 2b - c)(a + 2b + c)\)  (ii) \((a - b - c)^2\)

**Answer:**
(i) \(a^2 + 4ab + 4b^2 - c^2\)  
(ii) \(a^2 + b^2 + c^2 - 2ab - 2ac + 2bc\)

**A Great Website for More Detail:**

[www.k12connect.ca/~steve_soames/Lessons%20&%20Formulas/EXPRESSION/MultBin oTrino.htm](http://www.k12connect.ca/~steve_soames/Lessons%20&%20Formulas/EXPRESSION/MultBin oTrino.htm)
4) Expanding \((a \pm b)^3\)

**Problem:** Expand: (i) \((a + b)^3\)  (ii) \((a - b)^3\)

**Solution:**

(i) \((a + b)^3 = (a + b)^2(a + b) = (a^2 + 2ab + b^2)(a + b)\)
\[= a^3 + a^2b + 2a^2b + 2ab^2 + b^2a + b^3 = a^3 + 3a^2b + 3ab^2 + b^3\]

(ii) \((a - b)^3 = (a - b)^2(a - b) = (a^2 - 2ab + b^2)(a - b)\)
\[= a^3 - a^2b - 2a^2b + 2ab^2 + b^2a - b^3 = a^3 - 3a^2b + 3ab^2 - b^3\]

**Notes:** \(ab^2\) and \(b^2a\) are "like" terms which is why \(2ab^2 + b^2a = 3a^2b\).

**Common error:** In (i) \((a + b)^3 = a^3 + b^3\)

**Practice:** Expand \((2x - 3y)^3\)

**Answer:** \(8x^3 - 36x^2y + 54xy^2 - 27y^3\)

**A Great Website for More Detail:** [purplemath.com/modules/binomial.htm](http://purplemath.com/modules/binomial.htm)
5) Factoring Easy Trinomials

Problem: Factor:  
(i) \(x^2 + 5x + 4\)  
(ii) \(x^2 + 3x - 4\)  
(iii) \(6x^2 + 17x + 5\)  
(iv) \(6x^2 - 13x - 5\)

Solution:

(i) \(x^2 + 5x + 4 = (x + 4)(x + 1)\)

(ii) \(x^2 + 3x - 4 = (x + 4)(x - 1)\)

(iii) \(6x^2 + 17x + 5 = (3x + 1)(2x + 5)\)

(iv) \(6x^2 - 13x - 5 = (3x + 1)(2x - 5)\)

Notes:  
\[(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd\]
We need to find \(a\) and \(c\) for the \(x^2\) coefficient and \(b\) and \(d\) for the constant so that \(ad + bc\) is the \(x\) coefficient. We do this largely by trial and error. When stuck, there is a sure method. See the next question.

Common error: In (ii) \(x^2 + 3x - 4 = (x - 4)(x + 1)\)

Practice: Factor:  
(i) \(x^2 - 2x - 15\)  
(ii) \(9x^2 + 12x + 4\)

Answer:  
(i) \((x - 5)(x + 3)\)  
(ii) \((3x + 2)^2\)

A Great Website for More Detail: themathpage.com/alg/factoring-trinomials.htm
6) Factoring Less Easy Trinomials Using the Quadratic Formula

Problem: Factor: (i) $x^2 + 3x + 1$  (ii) $6x^2 - 5x - 2$

Hint: EVERY quadratic expression of the form $ax^2 + bx + c$ can be factored as $a(x - r_1)(x - r_2)$, where $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Solution:

(i) If we solve $x^2 + 3x + 1 = 0$ using the quadratic formula with $a = 1$, $b = 3$, and $c = 1$, we find roots:
$r_1 = \frac{-3 + \sqrt{9 - 4(1)(1)}}{2} = \frac{-3 + \sqrt{5}}{2}$ and $r_2 = \frac{-3 - \sqrt{9 - 4(1)(1)}}{2} = \frac{-3 - \sqrt{5}}{2}$

$\therefore x^2 + 3x + 1 = (x - r_1)(x - r_2) = \left(x - \frac{-3 + \sqrt{5}}{2}\right)\left(x - \frac{-3 - \sqrt{5}}{2}\right) = \left(x + \frac{3 + \sqrt{5}}{2}\right)$

(ii) If we solve $6x^2 - 5x - 2 = 0$ using the quadratic formula with $a = 6$, $b = -5$, and $c = -2$, we find roots:
$r_1 = \frac{5 + \sqrt{25 - 4(6)(-2)}}{12} = \frac{5 + \sqrt{73}}{12}$ and $r_2 = \frac{5 - \sqrt{25 - 4(6)(-2)}}{12} = \frac{5 - \sqrt{73}}{12}$

$\therefore 6x^2 - 5x - 2 = 6(x - r_1)(x - r_2) = 6\left(x - \frac{5 + \sqrt{73}}{12}\right)\left(x - \frac{5 - \sqrt{73}}{12}\right)$

Notes: Note in (ii) that we put the original $x^2$ coefficient, 6, "out front" in the answer. We really reduced the problem to factoring $6\left(x - \frac{5}{6}\right)\left(x - \frac{1}{3}\right)$ but working with integers for $a, b$ and $c$ is easier than working with fractions.

Common error: In (ii) $x^2 + 3x - 4 = (x - 4)(x + 1)$

Practice: Factor: (i) $x^2 - 2x - 15$  (ii) $9x^2 + 12x + 4$

Answer: (i) $(x - 5)(x + 3)$  (ii) $(3x + 2)^2$

A Great Website for More Detail: [purplemath.com/modules/quadform.htm](http://purplemath.com/modules/quadform.htm)
7) Factoring difference of squares: \( a^2 - b^2 = (a - b)(a + b) \)

**Problem:** 1) Factor: (i) \( x^2 - 9 \)  (ii) \( x^4 - (y+1)^2 \)

2) Rationalize the denominator: \( \frac{1}{\sqrt{x} - 4} \)

**Solution:**
1)(i) \( x^2 - 9 = (x - 3)(x + 3) \)

(ii) \( x^4 - (y+1)^2 = (x^2 - (y+1))(x^2 + (y+1)) = (x^2 - y-1)(x^2 + y+1) \)

2) \( \frac{1}{\sqrt{x} - 4} = \frac{\sqrt{x} + 4}{(\sqrt{x} - 4)(\sqrt{x} + 4)} = \frac{\sqrt{x} + 4}{x - 16} \)

**Note:** The order of the factors doesn’t matter: \( a^2 - b^2 = (a - b)(a + b) = (a + b)(a - b) \)

**Common error:** \( x^2 - 9 = (x - 3)^2 \)

**Practice:** (i) Factor: \( 25x^2 - 16y^2 \)  (ii) Rationalize the denominator: \( \frac{2}{\sqrt{a + b}} \)

**Answer:** (i) \( (5x - 4y)(5x + 4y) \)  (ii) \( \frac{2(\sqrt{a} - b)}{a - b^2} \)

A Great Website for More Detail: [purplemath.com/modules/specfact.htm](http://purplemath.com/modules/specfact.htm)
8) Factoring difference of cubes: \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \)

**Problem:**
(i) Factor: \( 8x^3 - 27 \)

(ii) Rationalize the denominator: \( \frac{1}{\sqrt[3]{x} - 2} \)

**Solution:**

(i) \( 8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9) \)

(ii) \[
\frac{1}{x^3 - 2} = \frac{1}{(x^3 - 2)(x^3 + 2x^2 + 4)} = \frac{2x^3 + 2x}{x^3 + 2x^2 + 4} \cdot \frac{1}{x - 8}
\]

**Note:** The order of the factors doesn’t matter:

\( a^3 - b^3 = (a - b)(a^2 + ab + b^2) = (a^2 + ab + b^2)(a - b) \)

**Common error:** \( x^3 - 9 = (x - 3)(x^2 + 3) \)

**Practice:**
(i) Factor: \( x^6 - 64y^3 \)

(ii) Rationalize the denominator: \( \frac{2}{a^3 - b^3} \)

**Answer:**
(i) \( x^6 - 64y^3 = (x^2 - 4y)(x^4 + 4x^2y + 16y^2) \)

(ii) \[
\frac{2}{a^3 - b^3} = \frac{2(a^3 + a^3b^3 + b^3)}{a - b}
\]

**A Great Website for More Detail:** [purplemath.com/modules/specfact2.htm](http://purplemath.com/modules/specfact2.htm)
9) Factoring sum of cubes: \( a^3 + b^3 = (a + b)(a^2 - ab + b^2) \)

Problem: (i) Factor: \( 8x^3 + 27 \)

(ii) Rationalize the denominator: \( \frac{1}{\sqrt[3]{x} + 2} \)

Solution:

(i) \( 8x^3 + 27 = (2x + 3)(4x^2 - 6x + 9) \)

(ii) \( \frac{1}{x^3 + 2} = \frac{1}{x^3 + 2} \frac{\frac{2}{x^3} - \frac{1}{x^3} + 4}{\frac{2}{x^3} - \frac{1}{x^3} + 4} = \frac{\frac{2}{x^3} - \frac{1}{x^3} + 4}{x + 8} \)

Note: The order of the factors doesn’t matter:

\( a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a^2 - ab + b^2)(a + b) \)

Common error: \( x^3 - 9 = (x - 3)(x^2 + 3) \)

Practice: (i) Factor: \( x^6 + 64y^3 \) (ii) Rationalize the denominator: \( \frac{2}{a^3 + b^3} \)

Answer: (i) \( x^6 + 64y^3 = (x^2 + 4y)(x^4 - 4x^2y + 16y^2) \)

(ii) \( \frac{2}{a^3 + b^3} = \frac{2(a^2 - ab + b^2)}{a + b} \)

A Great Website for More Detail: purplemath.com/modules/specfact2.htm
10) Factoring \( a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \ldots + ab^{n-2} + b^{n-1}) \)

**Problem:** Factor: \( x^5 - y^5 \)

**Solution:** \( x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \)

**Note:** This works for any positive integer \( n \). Note the terms on the right are ALL positive. The exponents on \( a \) start at \( n - 1 \) and decrease to 0 and the exponents on \( b \) start at 0 and increase to \( n - 1 \).

**Common error:** \( x^5 - y^5 = (x - y)(x^4 + y^4) \)

**Practice:** Factor: \( 16x^4 - y^4 \)

**Answer:** \( 16x^4 - y^4 = (2x - y)(8x^3 + 4x^2y + 2xy^2 + y^3) \)

**A Great Website for More Detail:** [mathforum.org/library/drmath/view/55601.html](http://mathforum.org/library/drmath/view/55601.html)
11) Factoring \( a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + \ldots - ab^{n-2} + b^{n-1}) \)
and \( n \) MUST BE ODD!

**Problem:** Factor: \( x^5 + y^5 \)

**Solution:** \( x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4) \)

**Note:** Expressions such as \( x^2 + y^2 \) and \( x^4 + y^4 \) (the exponent is even) DO NOT have a factor of \( (x + y) \). This method only works when \( n \) is an ODD positive integer.

**Common error:** \( x^5 + y^5 = (x + y)(x^4 + y^4) \)

**Practice:** Factor: (i) \( 32x^5 + y^5 \) (ii) \( 16x^4 + y^4 \) where one factor is \( (2x + y) \).

**Answer:** (i) \( 32x^5 + y^5 = (2x + y)(16x^4 - 8x^3y + 4x^2y^2 - 2xy^3 + y^4) \)
(ii) You cannot because the exponent is EVEN.

**A Great Website for More Detail:** [en.wikipedia.org/wiki/Factorization](en.wikipedia.org/wiki/Factorization)
12) The Factor Theorem: Part 1

Problem: Factor the expression \( x^3 - 4x^2 + x + 6 \).

Solution: To find a factor, substitute \( x = 1, -1, 2, -2, 3, -3, 6, -6 \) in other words, divisors of 6, into the expression. The Factor Theorem tells us that if \( x = a \) makes the expression equal to 0, then \( x - a \) is a factor, that is, \( x - a \) will divide evenly into \( x^3 - 4x^2 + x + 6 \). Then we can find the remaining factors without using the Factor Theorem again.

\( x = 1: \quad 1 - 4 + 1 + 6 = 4 \neq 0 \)
\( x = -1: \quad -1 - 4 - 1 + 6 = 0 = 0 \)

Therefore \( x + 1 \) is a factor. Now we divide \( x + 1 \) into \( x^3 - 4x^2 + x + 6 \) as the first step to finding the other factors.

\[
\begin{array}{c|ccc|ccc}
 & x^3 & -4x^2 & +x & +6 \\
\hline
x+1 & x^2 & -5x & +6 \\
\hline
\hline
& -x^3 & -x^2 & \\
\hline
& -5x^2 & +x & +6 \\
\hline
\hline
& & -5x & -5x \\
\hline
& & 6x & +6 \\
\hline
\hline
& & & -(6x+6) \\
\hline
& & & 0
\end{array}
\]

\[
\therefore \quad x^3 - 4x^2 + x + 6 = (x+1)(x^2 - 5x + 6) = (x+1)(x-2)(x-3)
\]

Note: Polynomial division is just like ordinary division. For example,

Common error: Not properly subtracting to get the remainder at each step in the polynomial division.

Practice: Use the Factor Theorem to factor \( x^3 - 7x^2 + 16x - 12 \).

Answer: \( x^3 - 7x^2 + 16x - 12 = (x-2)^2(x-3) \)

A Great Website for More Detail: purplemath.com/modules/factrthm.htm
13) The Factor Theorem: Part 2

Problem: Find all the rational roots of \(2x^3 - 5x^2 - 4x + 3\).

Solution: By the Factor Theorem, all rational roots must be of the form \(\frac{a}{b}\), where \(a\) is a divisor of 3 and \(b\) is a divisor of 2. So we will substitute \(x = \frac{a}{b}\), where \(a = 1, -1, 3, -3\) and \(b = 1, 2\). The possible rational roots are \(\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}\). In the original expression. If \(x = r\) makes the expression equal to 0, then \(x - r\) is a factor and \(r\) is a root. Since we have cubic, there are at most three rational factors.

\[
\begin{align*}
x &= 1: \quad 2 - 5 - 4 + 3 = -4 \neq 0 \\
x &= -1: \quad -2 - 5 + 4 + 3 = 0 \\
x &= 3: \quad 54 - 45 - 12 + 3 = 0 \\
x &= -3: \quad -54 + 45 + 12 + 3 = 0 \\
x &= \frac{1}{2}: \quad \frac{1}{4} - \frac{5}{4} - 2 + 3 = 0 
\end{align*}
\]

Since we have three roots and the original expression is a cubic polynomial, we are finished. The rational roots (and in fact all the roots) are \(-1, 3, \frac{1}{2}\).

Note: In factored form, \(2x^3 - 5x^2 - 4x + 3 = 2(x+1)(x-3)(x-\frac{1}{2}) = (x+1)(x-3)(2x-1)\).

Common error: Here, it is very easy to make arithmetic errors when substituting numbers like \(\frac{1}{2}\) into the expression.

Practice: Use the Factor Theorem to find the rational roots of \(2x^3 - 3x^2 - x - 2\).

Answer: \(x = 2\); the other two roots are complex numbers.

\(2x^3 - 3x^2 - x - 2 = (2x^2 + x + 1)(x - 2)\)

A Great Website for More Detail: purplemath.com/modules/factrthm.htm
14) Polynomial Division

**Problem:** Divide \( x^3 - 5x^2 + x + 6 \) by \( x - 4 \). Express your answer in the form \( \frac{x^3 - 5x^2 + x + 6}{x - 4} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}} \) both in the form \( x^3 - 5x^2 + x + 6 = \text{Quotient} \times \text{Divisor} + \text{Remainder} \).

**Solution:**

\[
\begin{array}{c|ccccc}
\text{x-4} & x^3 & -4x^2 & +x & +6 \\
\hline
& x^2 & -x & -3 & \\
\hline
& -x^3 & +5x^2 & -x & -6 \\
\hline
& 0 & 4x^2 & +2x & +6 \\
\hline
& 0 & 4x^2 & +12x & \\
\hline
& & & -3x & +6 \\
\hline
& & & & -3x & +12 \\
\hline
& & & & & -6
\end{array}
\]

Therefore, \( \frac{x^3 - 5x^2 + x + 6}{x - 4} = x^2 - x - 3 - \frac{6}{x - 4} \) and \( x^3 - 5x^2 + x + 6 = (x - 4)(x^2 - x - 3) - 6 \).

**Common error:** Not properly subtracting to get the remainder at each step in the polynomial division.

**Practice:** Simplify \( \frac{x^3 - 2x^2 - 3x + 3}{x + 2} \) and express your answer as in the example.

**Answer:** \[ \frac{x^3 - 2x^2 - 3x + 3}{x + 2} = x^2 - 4x + 5 - \frac{7}{x + 2} \]

\( x^3 - 2x^2 - 3x + 3 = (x^2 - 4x + 5)(x + 2) - 7 \)

**A Great Website for More Detail:** [purplemath.com/modules/factrthm.htm](http://purplemath.com/modules/factrthm.htm)
15) Solving for the Roots of a Polynomial

**Problem:** Solve: (i) $x^2 - 5x + 6 = 0$ (ii) $x^2 - 5x + 1 = 0$ (iii) $x^3 - 7x = -6$

**Solution:** (i) Factoring, $x^2 - 5x + 6 = (x - 3)(x - 2) = 0$ and so $x = 2$ or $x = 3$.

(ii) Using the quadratic formula with $a = 1$ and $b = -5$ and $c = 1$, we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(1)}}{2} = \frac{5 \pm \sqrt{21}}{2},$$

that is, the roots are $x = \frac{5 + \sqrt{21}}{2}$ and $x = \frac{5 - \sqrt{21}}{2}$.

(iii) Apply the Factor Theorem to $P(x) = x^3 - 7x + 6$:

$P(1) = 0 \quad P(1) = 12 \quad P(2) = 0 \quad P(-2) = 12 \quad P(3) = 12 \quad P(-3) = 0$

We can stop even though we haven't checked $P(6)$ nor $P(-6)$ (remember that we check all divisors of 6) because we KNOW a cubic polynomial has exactly three roots. So the roots are 1, 2, and $-3$.

**Note 1:** $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has roots $r_1, r_2, r_3, \ldots$, and $r_n \Leftrightarrow P(x) = a_n (x - r_1)(x - r_2)(x - r_3)\ldots(x - r_n)$

**Note 2:** $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has roots $r_1, r_2, r_3, \ldots$, and $r_n \Leftrightarrow$

the $x$ intercepts of $y = P(x)$ are the REAL numbers in the set \{r_1, r_2, r_3, \ldots, r_n\}

**Common errors:** (i) Getting the signs wrong when factoring a quadratic. (ii) Substituting incorrectly when using the quadratic formula. (iii) Not evaluating $P(a)$ correctly when using the Factor Theorem.

**Practice:** Solve: (i) $x^2 + 11x + 30 = 0$ (ii) $x^2 - 2x + 2 = 0$ (iii) $x^3 - x^2 + x - 1 = 0$

**Answer:** (i) $-5, -6$ (ii) $1+i, 1-i$ (iii) $1, i, -i$ (Remember, $i = \sqrt{-1}$.)

A Great Website for More Detail: [oakroadsystems.com/math/polysol.htm](http://oakroadsystems.com/math/polysol.htm)
16) Solving “Factored” Inequalities: Numerators Only

Problem: Solve (i) \((x+1)(x-1)(x-2) \geq 0\)  (ii) \((x+1)^2x(x-1)^3 > 0\)

Solution: (i) The only \(x\) values where the expression can change from \(+\) to \(-\) or \(-\) to \(+\) are \(-1, 1, \) and \(2\). So we analyze the sign of each factor in the expression on a number line using these values.

\[
\begin{array}{cccccc}
-\infty & -1 & 1 & 2 & +
\end{array}
\]

The solution in interval notation is \([-1,1] \cup [2,\infty)\).

(ii) The only \(x\) values where the expression can change from \(+\) to \(-\) or \(-\) to \(+\) are \(-1, 0,\) and \(1\). However, the exponent on \(x+1\) is even: \((x+1)^2 \geq 0\) always!

\[
\begin{array}{cccccc}
-\infty & -1 & 0 & 1 & +
\end{array}
\]

The solution in interval notation is \((-\infty,-1) \cup (-1,0) \cup (1,\infty)\).

Note 1: If one of the factors is, for example, \((x+1)^{1/3}\), then the term will change sign at \(x = -1\). If the factor were \((x+1)^{2/3}\), it would not change sign.

Note 2: What is the geometric significance of this kind of problem? In (i), you have found where the graph of \(y = (x+1)(x-1)(x-2)\) is on or above the \(x\) axis.

Common errors: (i) Getting the signs wrong when the exponent on a factor is even.

Practice: (i) \(-(x-2)^3x(x-3)^{1/5} \geq 0\)  (ii) \(-(x-2)^4x(x-3)^{2/5} \geq 0\)

Answer: (i) \((-\infty,0] \cup [2,3]\)  (ii) \((-\infty,0] \cup \{2,3\}\)

Note: Because the expression in (ii) equals 0 at \(x = 2\) and \(x = 3\) (as well as \(x = 0\)), 2 and 3 are part of the solution.

A Great Website for More Detail: purplemath.com/modules/ineqsolv.htm
17) Solving “Factored” Rational Inequalities:

Problem: Solve: (i) \( \frac{(x+1)(x-2)}{(x-1)} \geq 0 \) (ii) \( \frac{(x+1)^2(x-1)^3}{x} > 0 \)

Solution: (i) The only \( x \) values where the expression can change from + to – or – to + are –1, 1, and 2. So we analyze the sign of the each factor in the expression on a number line using these values.

\[
\begin{array}{ccccc}
    x < -1 & -1 < x < 1 & 1 < x < 2 & x > 2 \\
    (-)(-)(-) & (+)(-)(-) & (+)(+)(-) & (+)(+)(+) \\
\end{array}
\]

The question asks for values of \( x \) where the expression is “\( \geq 0 \)”. So values of \( x \) where the numerator is 0 work. Values of \( x \) where the denominator is 0 do not! We include \( x = -1 \) and \( x = 2 \) but not \( x = 1 \). The solution in interval notation is \([-1, 1) \cup [2, \infty)\).

(ii) The only \( x \) values where the expression can change from + to – or – to + are –1, 0, and 1. However, the exponent on \( x + 1 \) is even: \((x+1)^2 \geq 0\) always!

\[
\begin{array}{ccccc}
    x < -1 & -1 < x < 0 & 0 < x < 1 & x > 1 \\
    (+)(-)(-) & (+)(-)(-) & (+)(+)(-) & (+)(+)(+) \\
\end{array}
\]

The question asks for values of \( x \) where the expression is “\( > 0 \)”. So values of \( x \) where the numerator is 0 do not work. Values of \( x \) where the denominator is 0 never work! We exclude \( x = -1, 0, 1 \). The solution in interval notation is \((-\infty, -1) \cup (-1, 0) \cup (1, \infty)\).

Note 1: If one of the factors is, for example, \((x+1)^{1/4}\), then \( x \geq -1 \) because you can’t take an even root of a negative number.

Note 2: If one of the factors is, for example, \((x+1)^{1/3}\), then the term will change sign at \( x = -1 \). If the factor were \((x+1)^{2/3}\), it would not change sign.

Common errors: (i) Getting the signs wrong when the exponent on a factor is even.

Practice: (i) \( \frac{-(x-3)^{1/5}}{x(x-2)^3} \geq 0 \) (ii) \( \frac{-(x-3)^{2/5}}{(x-2)^4x} \geq 0 \)

Answer: (i) \((-\infty, 0) \cup (2, 3]\) (ii) \((-\infty, 0) \cup \{3\}\)

A Great Website for More Detail: [purplemath.com/modules/ineqrtnl.htm](http://purplemath.com/modules/ineqrtnl.htm)
18) Completing the Square

Problem: Complete the square: (i) \( x^2 - 8x + 25 \) (ii) \( 3x^2 + 36x - 17 \) (iii) \(-2x^2 + 3x + 1\)

Solution: Remember \((x + a)^2 = x^2 + 2ax + a^2\)

(i) \( x^2 - 8x + 25 = x^2 - 8x + 16 + 9 = (x - 4)^2 + 9 \)

(ii) \( 3x^2 + 36x - 17 = 3(x^2 + 12x + 36) - 17 = 3(x + 6)^2 - 108 = 3(x + 6)^2 - 125 \)

(iii) \(-2x^2 + 3x + 1 = -2(x^2 - \frac{3}{2}x + \frac{9}{16}) + \frac{9}{8} = -2\left(x - \frac{3}{4}\right)^2 + \frac{17}{8} \)

Note: When completing the square on the expression \(Ax^2 + Bx + C\), start this way: \(Ax^2 + Bx + C = A\left(x^2 + \frac{B}{A}x\right) + C\), that is, factor the \(A\) from the \(x^2\) and \(x\) terms, but not the constant.

Common error: Not “compensating” correctly

Practice: (i) \( x^2 - 4x - 3 \) (ii) \(-3x^2 + 5x + 1\)

Answer: (i) \((x - 2)^2 - 7\) (ii) \(-3\left(x - \frac{5}{6}\right)^2 + \frac{37}{12}\)

A Great Website for More Detail: purplemath.com/modules/sqrquad.htm
19) Adding and Subtracting Rational Expressions

**Problem:** By getting a common denominator, simplify the following expressions:

(i) \( \frac{1}{3x} - \frac{1}{2y} + \frac{1}{6z} \)  
(ii) \( \frac{2x+1}{x-1} - \frac{x+1}{x+2} - \frac{5x+4}{x^2+x-2} \)

**Solution:**

(i) \( \frac{1}{3x} - \frac{1}{2y} + \frac{1}{6z} \)

Get the lowest common denominator:

\[
\frac{2yz}{6xyz} - \frac{3xz}{6xyz} + \frac{xy}{6xyz} = \frac{2yz - 3xz + xy}{6xyz}
\]

(ii) \( \frac{2x+1}{x-1} - \frac{x+1}{x+2} - \frac{5x+4}{x^2+x-2} \)

Factor the denominators:

\[
\frac{2x+1}{x-1} - \frac{x+1}{x+2} - \frac{5x+4}{(x-1)(x+2)}
\]

Get the lowest common denominator:

\[
\frac{(2x+1)(x+2)}{(x-1)(x+2)} - \frac{(x+1)(x-1)}{(x-1)(x+2)} - \frac{5x+4}{(x-1)(x+2)}
\]

Expand the numerator:

\[
\frac{2x^2 + 5x + 2 - (x^2 - 1) - (5x + 4)}{(x-1)(x+2)} = \frac{x^2 - 1}{(x-1)(x+2)} = \frac{x+1}{x+2}
\]

Note: Adding and subtracting expressions uses exactly the same methods as adding and subtracting ordinary numerical fractions!

**Common error:** Not using the lowest common denominator which leads to a more complicated expression and more chances of making a mistake!

**Practice:**

(i) \( \frac{3}{2a} - \frac{1}{3b} + \frac{5}{6ab} \)  
(ii) \( \frac{1}{x^2} + \frac{x-2}{x(x+2)} - \frac{x}{(x+2)^2} \)

**Answer:**

(i) \( \frac{9b - 2a + 5}{6ab} \)  
(ii) \( \frac{x^2 + 4}{x^2(x+2)^2} \)

A Great Website for More Detail: [purplemath.com/modules/rtnladd.htm](http://purplemath.com/modules/rtnladd.htm)
20) Multiplying and Dividing Rational Expressions

**Problem:** Simplify the following rational expressions:

(i) \( \frac{(x^2 - 16)}{(x - 4)^3} \times \frac{(x^2 - 4x)^2}{x^3 + 64} \)

(ii) \( \frac{x^2 + 5xy + 4y^2}{x^2 + 4xy + 4y^2} \div \frac{x^2 + xy}{x^2 + 2xy} \)

**Solution:**

(i) \[
\frac{(x^2 - 16)}{(x - 4)^3} \times \frac{(x^2 - 4x)^2}{x^3 + 64}
= \frac{(x - 4)(x + 4)}{(x - 4)^3} \times \frac{x^2(x - 4)^2}{(x + 4)(x^2 - 4x + 16)}
= \frac{x^2}{x^2 - 4x + 16}
\]

(ii) \[
\frac{x^2 + 5xy + 4y^2}{x^2 + 4xy + 4y^2} \div \frac{x^2 + xy}{x^2 + 2xy}
= \frac{(x + 4y)(x + y)}{(x + 2y)^2} \div \frac{x(x + y)}{x(x + 2y)}
\]

Factor:
\[
= \frac{(x + 4y)(x + y)}{(x + 2y)^2} \times \frac{x(x + 2y)}{x(x + y)}
\]

Invert and multiply:
\[
= \frac{(x + 4y)(x + y)}{(x + 2y)^2} \times \frac{x(x + 2y)}{x(x + y)}
= \frac{x + 4y}{x + 2y}
\]

**Note:** When you multiply or divide rational expressions, you use exactly the same methods as adding and subtracting ordinary numerical fractions!

**Common errors:** Students often divide out “terms” instead of “factors”. For example, in the original question (i), \( x^2 \) is a term, while \( x^2 - 16 \) is a factor.

**Practice:** Simplify the following rational expressions:

(i) \( \frac{(x^2 + 7x + 6)^2}{(x^2 + 12x + 36)(x^2 - 1)} \times \frac{x^3 - 1}{x^2 + x + 1} \)

(ii) \( \frac{(z^3 + 5z^2)^3}{z^3 + 10z + 25} \div \frac{z^8(z^3 + 125)}{z^2 - 5z + 25} \)

**Answer:** (i) \( x + 1 \) (ii) \( \frac{1}{z^2} \)

A Great Website for More Detail: [purplemath.com/modules/rtnlmult.htm](http://purplemath.com/modules/rtnlmult.htm)
Solutions Part IV

Geometry

MP³ Section IV: Geometry 1
1) Pythagorean Theorem

**Problem:** In the following diagrams find the value of the unknowns:

(i) ![Pythagorean Theorem Diagram](image1.png)

(ii) ![Pythagorean Theorem Diagram](image2.png)

(iii) ![Pythagorean Theorem Diagram](image3.png)

**Solution:**

(i) \[ x^2 = 3^2 + 4^2 = 25 \quad \therefore x = 5 \]

(ii) \[ 3^2 + x^2 = 9 - 4 = 5 \quad \therefore x = \sqrt{5} \]

(iii) Let BC = y. Then \[ 5^2 = 25 = y^2 + 3^2 \quad \therefore y^2 = 25 - 9 = 16 \quad \therefore y = 4 \]

\[ AB = 10 - y = 10 - 4 = 6. \] Now we can find \( x \):

\[ x^2 = 3^2 + 6^2 = 45 \quad \therefore x = \sqrt{45} = 3\sqrt{5}. \]

**Note:** The important point is that if the length of any two sides of a right angled triangle is known we can determine the third side using the Pythagorean Theorem.

**Common error:** \[ x^2 = a^2 + b^2 \quad \therefore x = a + b \]

**Practice:**

Find \( x \) in the following:

(i) ![Practice Diagram](image4.png)

(ii) ![Practice Diagram](image5.png)

**Answer:**

(i) \( x = 2\sqrt{6} \)

(ii) \( x = 10 \)

A Great Website for More Details: [purplemath.com/modules/perimetr3.htm](http://purplemath.com/modules/perimetr3.htm)
2) Angles in a Triangle

**Problem:** Find the values of angles $x$ and $y$ (in degree measure) from the following diagrams:

(i) ![Diagram 1](image1.png)

(ii) ![Diagram 2](image2.png)

**Solution:**

(i) $x + 2x + 3x = 180°$. $\because 6x = 180° \therefore x = 30°$.

(ii) $x + y + 50° = 180°$. But $x = y$ (Isosceles Triangle!)

So $2x + 50° = 180°$ and $x = y = 65°$.

**Note:** Once you know any two angles of a triangle you know all three.

**Common error:** Making a mistake when solving the equation for the unknown.

**Practice:** Find $x$ in each of the following diagrams:

(i) ![Diagram 3](image3.png)

(ii) ![Diagram 4](image4.png)

**Answer:** (i) $x = 60°$  (ii) $x = 100°$

**A Great Website for More Details:** [mathopenref.com/triangleinternalangles.html](http://mathopenref.com/triangleinternalangles.html)
3) The Parallel Line Theorem

Problem: Find the values in degrees of $x$ and $y$ in the following diagrams:

(i) \[ \angle BDE = \angle ABC = 140^\circ \]

Solution: i) \( \angle ABC = \angle BDE = x \) (corresponding angles)
\[ \therefore x = 140^\circ \]
Also, \( x + y = 180^\circ \) and so \( y = 40^\circ \).

(ii) \( y = 3x - 20^\circ \) (alternate angles)
Also, \( 2x + y = 180^\circ \).
\[ \therefore 2x + 3x - 20^\circ = 180^\circ \Rightarrow 5x = 200^\circ \Rightarrow x = 40^\circ \]

So \( x = 40^\circ \) and \( y = 3(40) - 20 = 100^\circ \).

Note: When two lines are parallel, the sum of the interior angles on the same side of the transversal sum to 180°. So we could have solved (ii) by writing \( 2x + 3x - 20 = 180 \) and then finding first \( x \) and then \( y \).

Common error: Misidentification of corresponding and/or alternate angles.

Practice: Find the values in degrees of \( x \) and \( y \):

\[ \angle x = 60^\circ \]

Answer: \( x = 120^\circ , y = 60^\circ \)

A Great Website for More Detail:
sonoma.edu/users/w/wilsonst/Papers/Geometry/parallel/default.html

MP³ Section IV: Geometry 4
4) Congruent Triangles

**Problem:** Triangles I and II are congruent. Name the congruent triangles so that the vertices “correspond” and determine the values $x, y, u,$ and $v$.

![Diagram of triangles I and II](image)

**Solution:** Note that the two triangles are congruent by AAS: $\triangle ABC \cong \triangle ZYX$

\[x = 180 - 48 - 65 = 67\quad y = 180 - 48 - 65 = 67\quad u = \frac{180 - 48 - 65}{2} = 7\quad v = \frac{180 - 48 - 65}{2} = 5\]

**Notes:** When two congruent triangles are superimposed, every corresponding component matches up: sides, angles, area, perimeter, **everything**. Congruency can be determined by SSS, SAS, ASA, and AAS but **not by SSA nor by AAA**!

**Common error:** $\triangle ABC \cong \triangle XYZ$. This is wrong because the vertices do not correspond.

**Practice:** The two triangles below are congruent. Name the congruent triangles so that the vertices “correspond” and determine the values $a, b, c,$ and $d$.

![Diagram of triangles III and IV](image)

**Answer:** $\triangle ABC \cong \triangle PRQ \quad a = b = 60\quad c = 6\quad d = 4$

**A Great Website for More Detail:**

[sonoma.edu/users/w/wilsonst/Papers/Geometry/parallel/default.html](http://sonoma.edu/users/w/wilsonst/Papers/Geometry/parallel/default.html)
5) Similar Triangles

**Problem:** Triangles I and II are similar. Name the similar triangles so that the vertices “correspond” and determine the values $x, y, \text{ and } z$.

**Solution:** Note that the two triangles are similar by AA: $\triangle ABC \sim \triangle DEF$

$z = 180° - 80° - 60° = 40°$. By similar triangles, $\frac{x}{10} = \frac{8}{5} = \frac{4}{2}$

Solving gives $x = 8$ and $y = \frac{32}{5}$.

**Note:** Similarity can be determined by SSS, SAS, and AA (which is the same as AAA!) Here, when we use S, we are referring not to equality of sides but to the ratio of corresponding sides.

**Common error:** $\triangle ABC \sim \triangle EDF$. This is wrong because the vertices do not correspond.

**Practice:** The two triangles below are similar. Name the similar triangles so that the vertices “correspond” and determine the values $x$ and $y$.

**Answer:** $\triangle ABC \sim \triangle XYZ \quad x = 24 \quad y = \frac{39}{2}$

**A Great Website for More Detail:**

sonoma.edu/users/w/wilsonst/Papers/Geometry/parallel/default.html

MP$^3$ Section IV: Geometry 6
6) Area of a Triangle

Problem: Find the area of \( \triangle ABC \):

(i) \hspace{1cm} (ii)

Solution: (i) \( \text{Area } A = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} \times 8 \times 5 = 20 \text{ square units.} \)

(ii) \( \text{Area } A = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} \times 7 \times 4 = 14 \text{ square units.} \)

Note: Choose one side to be the base. Then the height is the length of the perpendicular from the vertex opposite to the base. Sometimes the foot of the perpendicular is on an extension of the base as in diagram (ii).

Common error: Not realizing the perpendicular height from B to AC is on an extension of AC.

Practice: Find the area of \( \triangle ABC \).
(Hint: first use the Pythagorean Theorem to find the perpendicular height.)

Answer: \( \text{Area} = \sqrt{3} \text{ sq. units.} \)

A Great Website for More Detail:

en.wikipedia.org/wiki/Triangle#Computing_the_area_of_a_triangle
7) **Area and circumference of a circle**

**Problem:** (i) Find the circumference and area of a circle of radius 8 cm.

(ii) Find the circumference and area of a circle of diameter 8 m.

(iii) If the area of a circle is \(16\pi\), find the radius.

**Solution:** (i) radius \(r = 8\) cm

Circumference \(C = 2\pi r = 2\pi \times 8 = 16\pi\) cm and area \(A = \pi r^2 = \pi \times 8^2 = 64\pi\) cm\(^2\).

(ii) Diameter \(d = 8\). \(C = \pi d = \pi \times 8 = 8\pi\) m and area \(A = \pi \times \frac{d^2}{4} = \pi \times \frac{8^2}{4} = 16\pi\) m\(^2\).

(iii) \(A = \pi r^2 = 16\pi\). Then \(r^2 = 16\) and \(r = 4\) m.

**Note:** Take any circle—ANY CIRCLE!—and wrap copies of the radius around the semi-circle. How many copies will it take? \(\pi \approx 3.14159\ldots\)

![Diagram of a circle with radius labeled](image)

**Common error:** Using the diameter value \(d\) in the formulas \(2\pi r\) and \(\pi r^2\); similarly using the radius in the formulas \(\pi d\) and \(\frac{\pi d^2}{4}\).

**Practice:** Find the circumference and area of a circle of diameter 12 units.

**Answer:** \(C = 12\pi\) units and \(A = 36\pi\) units\(^2\).

**A Great Website for More Detail:** [mathforum.org/dr.math/faq/faq.pi.html](http://mathforum.org/dr.math/faq/faq.pi.html)
8) Arc Length and Area of a Sector of a Circle

**Problem:** (i) State the arc length $s$ and the area $A$ of the sector of this circle. Assume $\theta$ is in **radian** measure.

(ii) In the circle below, find the arc length $s$ and the area $A$.

![Diagram of a circle with a sector and arc labeled](image)

**Solution:** (i) Arc length $s = r\theta$ units and $A = \frac{1}{2} r^2 \theta$ units$^2$. (Remember: the angle must be measured in radians.)

(ii) $s = r\theta = 6 \times \frac{\pi}{8} = \frac{3\pi}{4}$ cm; $A = \frac{1}{2} r^2 \theta = \frac{1}{2} (6)^2 \left(\frac{\pi}{8}\right) = \frac{9\pi}{4}$ cm$^2$

**Note:** In the correct use of the formulas in (i), $\theta$ must be in radians. If $\theta$ is given in degrees, convert to radians using this conversion formula:

\[
\text{angle in radians} = \theta \times \frac{\pi}{180}.
\]

**(VERY) Common error:** Using $\theta$ in degrees in the formulas in (i).

**Practice:** In the circle to the right, find the length $s$ corresponding sector area $A$.

(Hint: $120^\circ = \frac{2\pi}{3}$ radians)

![Diagram of a circle with a sector and arc labeled](image)

**Answer:** $s = \frac{20\pi}{3}$ m; $A = \frac{100\pi}{3}$ m$^2$

**A Great Website for More Details:**

[worsleyschool.net/science/files/sector/calculations.html](worsleyschool.net/science/files/sector/calculations.html)

**MP³ Section IV: Geometry 9**
9) **Volume of a Sphere, Box, Cone, Cylinder**

**Problem:** Find the volume $V$ of
(i) a sphere of radius $r = 3$ cm
(ii) a rectangular box with length $l = 8$ cm, width $w = 5$ cm, and height $h = 50$ cm
(iii) a right circular cone with height $h = 3$ cm and base radius $r = 0.04$ m
(iv) a circular cylinder with height $h = 0.2$ m and radius $r = 4$ cm.

**Solution:**

i) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (3^3) = 36\pi$ cm$^3$

(ii) $V = l \times w \times h = 8 \times 5 \times 50 = 2000$ cm$^3$

(iii) Get common units! $r = 0.04$ m = 4 cm

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 4^2 \times 3 = 16\pi$ cm$^3$

(iv) $h = 0.2$ m = 20 cm

$V = \pi r^2 h = \pi \times 4^2 \times 20 = 320\pi$ cm$^3$

**Note:** You have used a lot of volume formulas (like the cone and the sphere) as early as grade 3. In calculus, you finally get to **prove** them.

**Common error:** Not using a common measurement unit (such as cm) for all the variables in the volume formula.

**Practice:** Find the volume $V$ of
(i) a sphere of radius 2 cm

(ii) a box of length 0.3 m, width 7 cm, and height 3 cm

(iii) a cone of height 8 cm and radius 4 cm

(iv) a circular cylinder of height 12 cm and base radius 3 cm

**Answers:**

i) $\frac{32\pi}{3}$ cm$^3$ ii) 630 cm$^3$ iii) $\frac{128\pi}{3}$ cm$^3$ iv) $108\pi$ cm$^3$

**A Great Website for More Detail:** [en.wikipedia.org/wiki/Volume](en.wikipedia.org/wiki/Volume)
10) Angles of a Polygon

Problem: i) Find the sum of the interior angles of a pentagon. (ii) If all the interior angles of a pentagon are equal, how much is each interior angle?

Solution:

(i) The angles in a triangle sum to 180 degrees. Divide the pentagon into 3 triangles as shown. The sum of the interior angles of the pentagon is equal to $180 \times 3 = 540^\circ$

(ii) Since the 5 interior angles are equal each interior angle is equal to $\frac{540^\circ}{5} = 108^\circ$.

Note: By drawing lines from one vertex to each of the others in a polygon, you subdivide a polygon with 4 sides (a quadrilateral) into 2 triangles, a polygon with 5 sides (a pentagon) into 3 triangles, and, in general, a polygon with $n$ sides (an “n-gon”) into $n - 2$ triangles. The sum of the interior angles of an $n$-sided polygon is equal to $180(n - 2)^\circ$.

Common error: Dividing $180(n - 2)$ by $n$ to determine the size of an interior angle of an $n$-sided polygon. This is only correct when we know that all the $n$ interior angles of the $n$-sided polygon are equal, that is, the polygon is “regular”.

Practice: (i) Find the sum of the interior angles of a polygon with 9 sides. (ii) Find the interior angle of a regular 9-gon.

Answers: (i) $1260^\circ$ (ii) $140^\circ$

A Great Website for More Detail: regentsprep.org/regents/math/poly/LPoly1.htm
Solutions Part V

BASIC GRAPHS

\[ y = x^2 \quad y = x^4 \]

\[ y = x^{3/2} \quad y = x^{5/3} \]

MP³ Section V: Basic Graphs 1
1) Graphing \( y = x^n, \ n \in \mathbb{N} \)

**Problem:** (i) Graph the parabolas \( y = x^2 \) and \( y = x^4 \) on the same set of axes.
(ii) Graph the curves \( y = x^3 \) and \( y = x^5 \) on the same set of axes.

**Solution:**

(i) \( y = x^2 \)

(ii) \( y = x^3 \)

**Notes:** When \( n \) is even, \( y = x^n \) is symmetric in the \( y \) axis. That is why graphs symmetric in the \( y \) axis are called **even functions**. When \( n \) is odd, \( y = x^n \) is symmetric in the origin. That is why graphs symmetric in the origin are called **odd functions**.

**Common error:** Most of us realize \( y = x^4 \) is above \( y = x^2 \) when \( x > 1 \) but do not realize it is BELOW \( y = x^2 \) when \( 0 < x < 1 \).

**Practice:** Graph \( y = x^2 \) and \( y = x^3 \) on the same set of axes.

**Answer:**

**A Great Website for More Detail:**

[ugrad.math.ubc.ca/coursedoc/math100/notes/zoo/powers.html](ugrad.math.ubc.ca/coursedoc/math100/notes/zoo/powers.html) (requires JAVA)

**MP³ Section V: Basic Graphs 2**
2) Graphing $y = x^{m/n}$, $m, n \in \mathbb{N}$, $n$ Odd, and $m/n$ is a Reduced Fraction

Problem: (i) Graph the parabolas $y = x^{1/3}$ and $y = x^{2/3}$ on the same set of axes.
(ii) Graph the curves $y = x^{4/3}$ and $y = x^{5/3}$ on the same set of axes.

Solution:

(i) 

(ii) 

Notes: When $0 < m/n < 1$, the graph of $y = x^{m/n}$ looks like the square root or cube root function. When $m/n > 1$, the graph of $y = x^{m/n}$ looks like the parabola or the cubic.

Common error: Mixing up when $y = x^{m/n}$ is always positive and when it is sometimes $+$ and sometimes $-$. 

Practice: Graph $y = x^{3/5}$ and $y = x^{5/3}$ on the same set of axes.

Answer: 

A Great Website for More Detail: wmueller.com/precalculus/families/pwrfrac.html
3) Graphing $y = x^{m/n}$, $m, n \in \mathbb{N}$, $n$ Even, and $m/n$ is a Reduced Fraction

**Problem:** Graph the parabolas $y = x^{1/2}$ and $y = x^{3/2}$ on the same set of axes.

**Solution:**

![Graph of $y = x^{1/2}$ and $y = x^{3/2}$](image)

**Notes:** The domain of both of these functions is $\{x \in \mathbb{R} \mid x \geq 0\}$. When the exponent is between 0 and 1, the graph is concave down. When it is greater than 1, it is concave up.

**Common error:** Forgetting that $x$ must be $\geq 0$.

**Practice:** Graph $y = x^{1/4}$ and $y = x^{5/4}$ on the same set of axes.

**Answer:**

![Graph of $y = x^{1/4}$ and $y = x^{5/4}$](image)

A Great Website for More Detail: [ugrad.math.ubc.ca/coursedoc/math100/notes/zoo/powers2.html](https://ugrad.math.ubc.ca/coursedoc/math100/notes/zoo/powers2.html) (requires JAVA)
4) Graphing \( y = x^{-n} = \frac{1}{x^n}, \, n \in \mathbb{N} \)

**Problem:** Graph the parabolas \( y = x^{-1} = \frac{1}{x} \) and \( y = x^{-2} = \frac{1}{x^2} \) on the same set of axes.

**Solution:**

![Graph of \( y = \frac{1}{x} \) and \( y = \frac{1}{x^2} \)]

**Notes:** First, we know \( x \neq 0 \). When \( n \) is even, \( y = x^{-n} = \frac{1}{x^n} > 0 \).

When \( n \) is odd, \( y = x^{-n} = \frac{1}{x^n} > 0 \) when \( x > 0 \) and \( y = x^{-n} = \frac{1}{x^n} < 0 \) when \( x < 0 \).

**Common error:** Confusing \( y = x^{-n} = \frac{1}{x^n} \) with \( y = x^{1/n} \) (which I — the author — did twice while composing this darn question!)

**Practice:** Graph \( y = x^{-3} = \frac{1}{x^3} \) and \( y = x^{-4} = \frac{1}{x^4} \) on the same set of axes.

**Answer:**

![Graph of \( y = \frac{1}{x^3} \) and \( y = \frac{1}{x^4} \)]

**A Great Website for More Detail:** [wmueller.com/precalculus/families/pwrfrac.html](https://wmueller.com/precalculus/families/pwrfrac.html)
5) Graphing \( y = x^{\frac{1}{n}} = \frac{1}{x^n} \), \( n \in \mathbb{N} \)

**Problem:** Graph the parabolas \( y = \frac{1}{x^{1/2}} \) and \( y = \frac{1}{x^{1/3}} \) on the same set of axes.

**Solution:**

\[
\begin{align*}
\text{Graph} & \quad y = x^{\frac{1}{2}} \\
\text{Graph} & \quad y = x^{\frac{1}{3}} \\
\end{align*}
\]

**Notes:** First, we know \( x \neq 0 \). When \( n \) is even, \( y = \frac{1}{x^{1/n}} > 0 \).

When \( n \) is odd, \( y = \frac{1}{x^{1/n}} > 0 \) when \( x > 0 \) and \( y = \frac{1}{x^{1/n}} < 0 \) when \( x < 0 \).

**Common error:** Confusing \( y = x^{-n} = \frac{1}{x^n} \) with \( y = \frac{1}{x^{1/n}} \).

**Practice:** Graph \( y = \frac{1}{x^{1/4}} \) and \( y = \frac{1}{x^{1/5}} \) on the same set of axes.

**Answer:**

\[
\begin{align*}
\text{Graph} & \quad y = x^{\frac{1}{4}} \\
\text{Graph} & \quad y = x^{\frac{1}{5}} \\
\end{align*}
\]

A Great Website for More Detail: [wmueller.com/precalculus/families/pwrfrac.html](http://wmueller.com/precalculus/families/pwrfrac.html)
6) Transformations (New Graphs from a Given Graph)

**Problem:** Given \( y = f(x) = x^2 \), graph and describe each of the following relative to \( f \).
(i) \( y = f(x) + 2 \)  (ii) \( y = f(x) - 2 \)  (iii) \( y = f(x + 2) \)  (iv) \( y = f(x - 2) \)  (v) \( y = f(2x) \)  (vi) \( y = 2f(x) \)

**Solution:**
(i) Shifts \( f \) up 2 units.
(ii) Shifts \( f \) down 2 units.
(iii) Shifts \( f \) left 2 units.
(iv) Shifts \( f \) right 2 units.
(v) Squeezes \( f \) by a factor of 2 units.
(vi) Stretches \( f \) by a factor of 2 units.

**Note:** Use this for graphs of every variety—trig, logs, exponents, more!

**Common error:** Thinking \( f(x + 2) \) shifts 2 units to the right because we added 2.

**Practice:** Given \( y = f(x) = \sin(x) \), describe each of the following relative to \( f \).
(i) \( y = f(x + \pi) \)  (ii) \( y = f(\pi x) \)  (iii) \( y = f(x) + \pi \)
(iv) \( y = f(x - \pi) \)  (v) \( y = \pi f(x) \)  (vi) \( y = f(x) - \pi \)

**Answer:** (i) Shifts \( f \) left \( \pi \) units.  (ii) Squeezes \( f \) by a factor of \( \pi \).
(iii) Shifts \( f \) up \( \pi \) units. (iv) Shifts \( f \) right \( \pi \) units. (v) Stretches \( f \) by a factor of \( \pi \) units.
(vi) Shifts \( f \) down \( \pi \) units.

**A Great Website for More Detail:**

people.hofstra.edu/Stefan_Waner/calctopic1/scaledgraph.html
Solutions Part VI

Solving

\[
ax + b = 0 \iff x = -\frac{b}{a}
\]

\[
ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
ax^3 + bx^2 + cx + d = 0 \iff x = \text{It gets complicated!}
\]

\[
ax^4 + bx^3 + cx^2 + dx + e = 0 \iff
x = \text{It gets even more complicated!}
\]

\[
ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \iff
x = \text{There is no formula!}
\]

Equations

MP³ Section VI: Solving Equations 1
1) Solving Linear Equations in One Variable

**Problem:** Solve each of the following equations:

(i) \(4x + 20 = 2 - 5x\)  
(ii) \(8(x - 4) + x = 6(x - 5) - (1 - x)\)  
(iii) \(\frac{x}{3} - \frac{2x}{5} + \frac{1}{30} = \frac{7}{10}\)

**Solution:**

(i) \(4x + 20 = 2 - 5x\)  
   \[\Leftrightarrow\]  
   \[9x = -18\]  
   \[\Leftrightarrow\]  
   \[x = -2\]

(ii) \(8(x - 4) + x = 6(x - 5) - (1 - x)\)  
   \[\Leftrightarrow\]  
   \[9x - 32 = 7x - 31\]
   \[\Leftrightarrow\]  
   \[2x = 1\]  
   \[\Leftrightarrow\]  
   \[x = \frac{1}{2}\]

(iii) \(\frac{x}{3} - \frac{2x}{5} + \frac{1}{30} = \frac{7}{10}\)  
   \[\Leftrightarrow\]  
   \[10x - 12x + 1 = 21\]
   \[\Leftrightarrow\]  
   \[-2x = 20\]  
   \[\Leftrightarrow\]  
   \[x = -10\]

**Common error:** Messing up one of the cardinal rules of math: “What you do to one side, you do to the other!”

**Practice:** Solve each of the following equations:

(i) \(7(4x - 5) = 8(3x - 5) + 9\)  
(ii) \(\frac{x - 1}{2} + \frac{2x + 1}{5} = 6\)

**Answer:**

(i) \(x = 1\)  
(ii) \(x = 7\)

**A Great Website for More Details:**
[wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut14_lineareq.html](http://wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut14_lineareq.html)
2) Solving Linear Inequalities

**Problem:** Solve the following inequalities:

(i)  $4x - 5 \leq 2x + 9$
(ii) $2x + 7 > 5x - 1$
(iii) $x - 4 < x + 6$

**Solution:**

(i) $4x - 5 \leq 2x + 9$ \[\iff \] $2x \leq 14$ \[\iff \] $x \leq 7$ \[\implies \] $x \in [7, \infty)$

(ii) $2x + 7 > 5x - 1$ \[\iff \] $-3x > -8$ \[\iff \] $x < \frac{8}{3}$ \[\implies \] $x \in \left(-\infty, \frac{8}{3}\right)$

(iii) $x - 4 < x + 6$ \[\iff \] $0 < 10$ This is always true! \[\implies \] $x \in \mathbb{R}$

**Note:** Solve linear inequalities just as you would a linear equation, except when you multiply or divide by a “$-$”. Then you must change “$<$” to “$>$” and vice-versa!

**Common error:** Not paying attention to the note!

**Practice:** Solve the following inequalities:

(i) $5(x - 3) \leq 3(x - 4)$
(ii) $4x - 3 < -7 + 4x$

**Answer:**

(i) $x \in \left(-\infty, \frac{3}{2}\right)$
(ii) No solution (ie., $\emptyset$)

A Great Website for More Details: purplemath.com/modules/ineqlin.htm
3) Solving Two Linear Equations in Two Variables

**Problem:** Solve for \( x \) and \( y \). (E1 and E2 refer to equation 1 and equation 2.)

(i) E1: \( x + 2y = -1 \)  
(ii) E1: \( x - 2y = 6 \)  
(iii) E1: \( x - 2y = 6 \)  

\[ \begin{align*}
E2: & \quad 5x - 2y = 7 \\
E2: & \quad 3x - 6y = 18 \\
E2: & \quad 3x - 6y = 3
\end{align*} \]

**Solution:**

(i) Add E1+E2: \( 6x = 6 \Leftrightarrow x = 1 \) Substitute \( x = 1 \) into E1:
\( 1 + 2y = -1 \Leftrightarrow 2y = -2 \Leftrightarrow y = -1 \) \( : \) The solution is \( (x, y) = (1, -1) \).

(ii) Multiply 3\times E1 = E3: \( 3x - 6y = 18 \)
Subtract E3-E2: \( 0 = 0 \) This is ALWAYS true! From E1: \( x = 6 + 2y \)
\( \therefore \) Solutions are \( \{(6 + 2y, y) \mid y \in \mathbb{R}\} \)

(iii) Multiply 3\times E1 = E3: \( 3x - 6y = 18 \)
Subtract E3-E2 =E4: \( 0 = 15 \) This is NEVER true!
\( \therefore \) There are no solutions.

**Notes:** There are LOTS of ways to solve these problems. Each equation represents a line. In (i), the lines are non-parallel and meet in a single point. In (ii), the lines are “coincident”. In (iii), they are parallel and never meet.

**Common error:** Making an arithmetic mistake when forming a new equation from the given equations.

**Practice:** Solve \( 3x - 2y = 7 \) and \( 2x - 5y = 12 \)

**Answer:** \( (x, y) = (1, -2) \)

**A Great Website for More Details:**
wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut49_systwo.html

MP³ Section VI: Solving Equations 4
4) Solving Quadratic Equations

**Problem:** Solve the following quadratic equations:
(i) \( x^2 - 5x + 6 = 0 \)  (ii) \( 3x^2 - 7x + 2 = 0 \)

**Solution:**
(i) \( x^2 - 5x + 6 = 0 \) \( \iff (x - 2)(x - 3) = 0 \) \( \iff x = 2 \) or \( x = 3 \)

(ii) \( 3x^2 - 7x + 2 = 0 \)

Here, it is easiest to use the quadratic formula: 
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(2)}}{2(3)} = \frac{7 \pm \sqrt{25}}{6}
\]

\[\therefore x = \frac{12}{6} = 2 \quad \text{or} \quad x = \frac{2}{6} = \frac{1}{3}
\]

**Note:** In (ii), if you factored, you would have found \( 3x^2 - 7x + 2 = (3x - 1)(x - 2) = 0 \), which again (thank the math gods) gives \( x = \frac{1}{3} \) or \( x = 2 \).

**Common error:** Factoring incorrectly.

**Practice:** Solve the quadratic equations: (i) \( x^2 - 3x - 10 = 0 \)  (ii) \( 6x^2 - 7x - 3 = 0 \)

**Answer:**
(i) \( x = -2, 5 \)  (ii) \( x = -\frac{1}{3}, \frac{3}{2} \)

**A Great Website for More Details:** [purplemath.com/modules/solvquad.htm](http://purplemath.com/modules/solvquad.htm)
5) Solving Equations Involving Square Roots

Problem: Solve the following equations for $x$:

(i) $\sqrt{x-2} = 5$ (ii) $\sqrt{4-3x} = x + 12$ (iii) $\sqrt{1+2x} - \sqrt{x} = 1$

Solution: (i) $\sqrt{x-2} = 5 \Rightarrow x - 2 = 25 \Rightarrow x = 27$

Check: Substitute $x = 27$ in the original equation:
Left Side $= \sqrt{27-2} = \sqrt{25} = 5$ Right Side $= 5 \therefore x = 5$

(ii) $\sqrt{4-3x} = x + 12 \Rightarrow 4 - 3x = x^2 + 24x + 144$

Move "everything" to one side and solve.

$\Rightarrow x^2 + 27x + 140 = 0 \Rightarrow (x + 20)(x + 7) = 0 \Rightarrow x = -20$ or $x = -7$

Check: Substitute $x = -20$ in the original equation:
Left Side $= \sqrt{4 + 60} = 8$ Right Side $= -20 + 12 = -8 \therefore x = -20$ is NOT a solution.

Check: Substitute $x = -7$ in the original equation:
Left Side $= \sqrt{4 + 21} = 5$ Right Side $= -7 + 12 = 5 \therefore x = -7$ is the only solution.

(iii) $\sqrt{1+2x} - \sqrt{x} = 1 \Rightarrow \sqrt{1+2x} = 1 + \sqrt{x} \Rightarrow 1 + 2x = 1 + 2\sqrt{x} + x$

Isolate the more complicated square root on one side. Square both sides.

$\Rightarrow 2\sqrt{x} = x \Rightarrow 4x = x^2 \Rightarrow x^2 - 4x = x(x - 4) = 0 \therefore x = 0$ or $x = 4$

Check: Exercise for you but both $x = 0$ and $x = 4$ work!

Note: When we square both sides of an equation, we may introduce solutions to the new equation that are not solutions to the original. For example, $x = -3 \Rightarrow x^2 = 9$.

The new equation also has $x = 3$ as a solution, but this is not a solution of the original equation, $x = -3$. By squaring, we introduced an "extraneous" solution.

Common error: Forgetting to do a left side/right check side for extraneous roots.

Practice: Solve the following equations: (i) $\sqrt{x+4} = 6$ (ii) $\sqrt{x-3} = x - 5$.

Answer: (i) $x = 32$ (ii) $x = 7$

A Great Website for More Detail: purplemath.com/modules/solverad.htm

MP³ Section VI: Solving Equations 6
Solutions Part VII

GRAPHING SECOND

ORDER RELATIONS
1) The Parabola

**Problem:** Graph the following parabolas and identify the vertex and the axis of symmetry: (i) \( y = x^2 \)  \( (ii) \ y = 2(x+1)^2 - 3 \)

**Solution:** (i)  \( \text{Vertex: (0,0); Axis of Symmetry: } x = 0 \)  \( \text{(ii) Vertex: (-1,3); Axis of Symmetry: } x = -1 \)

**Note:** In \( y = a(x - b)^2 + c \), the vertex is \((b, c)\) and the axis of symmetry is \(x = b \).
When \( a > 0 \), the parabola opens up; when \( a < 0 \), the parabola opens down.

**Common error:** In (ii), identifying the vertex as \((1, -3)\).

**Practice:** Graph \( y = -2(x-1)^2 + 3 \) and identify the vertex and the axis of symmetry.

**Answer:**  \( \text{Vertex: (1,3) } \text{Axis of Symmetry: } x = 1 \)

**A Great Website for More Detail:**
tutorial.math.lamar.edu/Classes/Alg/Parabolas.aspx

MP³ Section VII: Graphing Second Order Relations 2
2) The Circle

**Problem:** Graph the following circles and identify the radius and the centre:
(i) $x^2 + y^2 = 4$  
(ii) $(x - 1)^2 + (y + 2)^2 = 9$

**Solution:**

(i) $r = 2; \text{ Centre}(0,0)$

(ii) $r = 3; \text{ Centre}(1,-2)$

**Note:** In $(x - a)^2 + (y - b)^2 = r^2$, the centre is $(a,b)$ and the radius, of course, is $r$.

**Common error:** In (ii), identifying the centre as $(-1,2)$.

**Practice:** Graph $x^2 + (y + 0.5)^2 = 1$ and identify the radius and the centre.

**Answer:**

$r = 1$  
Centre $(0, -0.5)$

A Great Website for More Detail:
[wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut29_circles.html](http://wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut29_circles.html)
3) The Ellipse

**Problem:** Graph the following ellipses and identify the centre and the major and minor axes: (i) \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) (ii) \( \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1 \)

**Solution:**

(i) Centre (0,0); Major Axis: \( y \) axis (ie., \( x = 0 \))

Minor Axis: \( x \) axis (ie., \( y = 0 \))

(ii) Centre (1, -2); Major Axis: \( x \) axis

Minor Axis: \( y \) axis

Note: In \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), the \( x \) intercepts are \( \pm a \) and the \( y \) intercepts are \( \pm b \).

**Common error:** Confusing the major and minor axes.

**Practice:** Graph \( \frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1 \) and identify the centre and the major and minor axes:

**Answer:**

Centre (−1, 2)

Major Axis: \( x = -1 \); Minor Axis: \( y = 2 \)

A Great Website for More Detail:

4) The Hyperbola

Problem: Graph the following hyperbolas and identify the centre and intercepts:

(i) \( \frac{x^2}{4} - \frac{y^2}{9} = 1 \)  
(ii) \( \frac{y^2}{9} - \frac{x^2}{4} = 1 \)

Solution: (i) Centre (0,0)  
intercepts \( x = \pm 2 \); no \( y \) intercepts  
(ii) Centre (0,2)  
noc \( x \) intercepts; \( y \) intercepts = \( \pm 3 \)

Note: \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) is a hyperbola opening on the \( x \) axis.

\( \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \) is a hyperbola opening on the \( y \) axis.

Common Error: Putting the intercepts on the wrong axis.

Practice: Graph  
(i) \( x^2 - y^2 = 1 \)  
(ii) \( y^2 - x^2 = 1 \).

Answer: 
(i) Centre (0,0)  
intercepts \( x = \pm 2 \); no \( y \) intercepts  
(ii) Centre (0,2)  
noc \( x \) intercepts; \( y \) intercepts = \( \pm 3 \)

A Great Website for More Detail:

tutorial.math.lamar.edu/Classes/Alg/Hyperbolas.aspx
Solutions Part VIII

TRIGONOMETRY

\[ \sin(\theta) = \frac{O}{H} = \frac{4}{5} \quad \cos(\theta) = \frac{A}{H} = \frac{3}{5} \quad \tan(\theta) = \frac{O}{A} = \frac{4}{3} = \frac{\sin(\theta)}{\cos(\theta)} \]

\[ \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{5}{4} \quad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{5}{3} \quad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{3}{4} = \frac{\cos(\theta)}{\sin(\theta)} \]

\( \theta = \frac{\pi}{4} \)

\( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \)

MP³ Section VIII: Trigonometry
1) Angles in Standard Position: Degree Measure

Problem: (i) Draw in standard position the following angles:
(a) 30°  (b) 225°  (c) −80°  (d) −190°  (e) 390°

(ii) Give all angles, in degrees, which are “co-terminal” with (a) 30°  (b) −190°.

Solution:
(i) (a)  (b)  (c)  (d)  (e)

(ii) (a) All angles co-terminal with 30° are 30° + 360°k, where k ∈ Z.
(b) All angles co-terminal with −190° are −190° + 360°k, where k ∈ Z.

Note: Co-terminal angles differ by multiples of 360°.

Common error: Regarding co-terminal angles as equal. 30° and 390° are different angles whose terminal arms, in standard position, are the same. Here is an analogy: Consider the function f(x) = x². f(−3) = f(3) = 9 but that doesn’t make −3 = 3!

Practice: (i) In what quadrant is the terminal arm of −660°?
(ii) Give all the angles in degree measure co-terminal with −660°.

Answer: (i) First quadrant, since −660° is co-terminal with 60°
(ii) −660° + 360°k, where k ∈ Z.

A Great Website for More Details: themathpage.com/aTrig/measure-angles.htm#stand

MP³ Section VIII: Trigonometry 2
2) The Meaning of $\pi$

**Problem:** In the circle below with radius $r$, you can “fit” three and a “little portion more” of a radius around half the circle.

![Circle diagram](image)

We give a name to this number of radii. (i) The name is _____.

(ii) So the length of a half circle is given by ______.

(iii) This is why the circumference is given by _____.

**Solution:** (i) $\pi$ (ii) $\pi r$ (iii) $2\pi r$

**Note:** The most commonly used approximations for $\pi$ are $\frac{22}{7}$ or 3.14. An approximation to $\pi$, accurate to 7 decimals, is 3.1415926.

**Common problem (as opposed to error):** Not having a clue that $\pi$ is the number of times the radius of a circle wraps around a semi-circle!

**Practice:** The number $\pi$ plays a role in the calculation of certain areas. For example, the surface area of a sphere of radius $r$ is $4\pi r^2$ (4 times the area of a circle of radius $r$). Find the surface area of a sphere of radius 3.

**Answer:** $36\pi$

**A Great Website for More Detail:** [en.wikipedia.org/wiki/Pi](http://en.wikipedia.org/wiki/Pi)
3) Angles in Standard Position: Radian Measure

Problems: 1) Draw in standard position the following angles:
   (i) $\pi/6$  (ii) $5\pi/4$  (iii) $-4\pi/9$.
2) Give all angles, in radians, which are “co-terminal” with $\pi/6$ radians.

Solutions: 1)(i)  (ii)  (iii)

2) All angles co-terminal with $\pi/6$ are $\pi/6 + 2\pi k$, where $k \in \mathbb{Z}$.

Note: 1 radian is about 57°. This is just what you should expect since a little more than 3.14 radians, that is, $\pi$ radians, equals about $3.14 \times 57 \approx 180°$.

Common error: The formulas for arc length and area of a sector of a circle are $r\theta$ and $\frac{1}{2}r^2\theta$, respectively. Students sometimes substitute $\theta$ in degree measure. It must be in radians!

Practice: (i) In what quadrant is the terminal arm of $-7\pi/4$?
(ii) Give all the angles in degree measure co-terminal with $-7\pi/4$.

Answer: (i) First quadrant, since $-7\pi/4$ is co-terminal with $\pi/4$.
(ii) $-7\pi/4 + 2\pi k$, where $k \in \mathbb{Z}$.

A Great Website for More Details: themathpage.com/aTrig/radian-measure.htm
4) Degrees to Radians

Problem: Express each of the following in radian measure:

(i) $25^\circ$     (ii) $-150^\circ$     (iii) $1060^\circ$ (Remember 180 degrees $= \pi$ radians)

Solution: (i) $25^\circ = \frac{\pi}{180^\circ} \times 25^\circ = \frac{5\pi}{36}$ radians.

(ii) $-150^\circ = \frac{\pi}{180^\circ} \times (-150^\circ) = -\frac{5\pi}{6}$ radians.

(iii) $1060^\circ = \frac{\pi}{180^\circ} \times 1060^\circ = \frac{53\pi}{9}$ radians.

Note: $1^\circ = \frac{\pi}{180}$ radians $\therefore x^\circ = x\left(\frac{\pi}{180}\right)$ radians

Common misinterpretation: We always use the degree symbol when measuring with degrees but we don’t always say radians when using radian measure. So, when measuring angles, “2”, “2 radians”, and “2” mean, respectively, “2 degrees”, “2 radians”, and “2 radians” (which is about 115 degrees!)

Practice: Express each of the following in radian measure:

(i) $-210^\circ$     (ii) $300^\circ$     (iii) $240^\circ$.

Answer: (i) $-\frac{7\pi}{6}$     (ii) $\frac{5\pi}{3}$     (iii) $\frac{4\pi}{3}$.

A Great Website for More Detail:

teacherschoice.com.au/Maths_Library/Angles/Angles.htm
5) Radians to Degrees

**Problem:** Express each of the following radian measures in degrees:

(i) \( \frac{4\pi}{9} \)  
(ii) \( \frac{2\pi}{5} \)  
(iii) \( -\frac{7\pi}{6} \)  
(iv) \( -\frac{4\pi}{3} \)  
(Remember \( \pi \) radians = 180 degrees)

**Solution:**
(i) \( \frac{4\pi}{9} \) radians = \( \frac{4\pi}{9} \times \frac{180^\circ}{\pi} = 80^\circ \).

(ii) \( \frac{2\pi}{5} \) radians = \( \frac{2\pi}{5} \times \frac{180^\circ}{\pi} = 72^\circ \).

(iii) \( -\frac{7\pi}{6} \) radians = \( -\frac{7\pi}{6} \times \frac{180^\circ}{\pi} = -210^\circ \).

(iv) \( -\frac{4\pi}{3} \) radians = \( -\frac{4\pi}{3} \times \frac{180^\circ}{\pi} = -240^\circ \).

**Note:** 1 radian = \( \frac{180^\circ}{\pi} = 57.3^\circ \) : x radians = \( x \left( \frac{180^\circ}{\pi} \right) \)

**Common misinterpretation:** We always use the degree symbol when measuring with degrees but we don’t always say radians when using radian measure. So, when measuring angles, “2”, “2 radians”, and “2” mean, respectively, “2 degrees”, “2 radians”, and “2 radians” (which is about 115 degrees!)

**Practice:** Express each of the following radian measures in degrees:

(i) \( \frac{4}{3} \)  
(ii) \( \frac{3\pi}{4} \)  
(iii) \( \frac{7\pi}{10} \)  
(iv) \( \frac{1}{4} \)

**Answer:** (i) 76.4°  (ii) 135°  (iii) 126°  (iv) 14.32°

**A Great Website for More Details:**

6) Relating Angles in Standard Position in Quadrants One and Two

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

Problem: (i) Below, the second quadrant angle $170^\circ$ is drawn in standard position. Find and illustrate the related first quadrant angle, using the interval $(0,90^\circ)$.

(ii) Below, the first quadrant angle $\frac{\pi}{6}$ is drawn in standard position. Find and illustrate the related second quadrant angle, using the interval $\left(\frac{\pi}{2},\pi\right)$.

Solution: (i) First quad. angle = $180 - 170 = 10^\circ$ (ii) Second quad. angle = $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Notes: If $\theta$ satisfies $\frac{\pi}{2} < \theta < \pi$, the corresponding first quadrant angle using $\left(0,\frac{\pi}{2}\right)$ is $\pi - \theta$.

If $\theta$ satisfies $0 < \theta < \frac{\pi}{2}$, the corresponding second quadrant angle using $\left(\frac{\pi}{2},\pi\right)$ is still $\pi - \theta$!

Common error: Confusing the second quadrant angle in standard position $170^\circ$ with the $10^\circ$ angle that $170^\circ$ makes with the negative x axis.

Practice: (i) Find the first quadrant angle relatives of (a) $150^\circ$ (b) $2\pi/3$.

(ii) Find the second quadrant relatives of (a) $75^\circ$ (b) $\pi/9$.

Answer: (i)(a) $30^\circ$ (b) $\pi/3$ (ii)(a) $105^\circ$ (b) $8\pi/9$

A Great Website for More Detail: oakroadsystems.com/twt/refangle.htm

MP³ Section VIII: Trigonometry 7
7) Relating Angles in Standard Position in Quadrants One and Three

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

Problem: (i) Below, the third quadrant angle 190° is drawn in standard position. Find and illustrate its related first quadrant angle, using the interval (0, 90°).

(ii) Below, the first quadrant angle π/6 is drawn in standard position. Find and illustrate its related third quadrant angle, using the interval (π/2, π).

Solution: (i) First quad. angle = 190° - 180° = 10° (ii) Third quad. angle = π/6 + π = 7π/6

Notes: If θ satisfies π < θ < 3π/2, the corresponding first quadrant angle using (0, π/2) is θ - π.

If θ satisfies 0 < θ < π/2, the corresponding third quadrant angle using (π, 3π/2) is π + θ!

Common error: Confusing the third quadrant angle in standard position 190° with the 10° angle that 190° makes with the negative x axis.

Practice: (i) Find the first quadrant angle relatives of (a) 250°  (b) 7π/6.

(ii) Find the third quadrant relatives of (a) 75°  (b) π/9.

Answer: (i)(a) 70°  (b) π/6  (ii)(a) 255°  (b) 10π/9

A Great Website for More Detail: oakroadsystems.com/twt/refangle.htm

MP³ Section VIII: Trigonometry 8
8) Relating Angles in Standard Position in Quadrants One and Four

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

**Problem:** (i) Below, the fourth quadrant angle $-10^\circ$ is drawn in standard position. Find and illustrate its related first quadrant angle, using the interval $(0, 90^\circ)$.

(ii) Below, the first quadrant angle $\frac{\pi}{6}$ is drawn in standard position. Find and illustrate its related fourth quadrant angle, using the interval $\left(-\frac{\pi}{2}, 0\right)$.

![Diagram](image1)

**Solution:** (i) First quad. angle $= -(-10) = 10^\circ$ (ii) Fourth quad. angle $= -\frac{\pi}{6}$

![Diagram](image2)

**Notes:** If $\theta \in \left(-\frac{\pi}{2}, 0\right)$, the corresponding first quadrant angle using $\left(0, \frac{\pi}{2}\right)$ is $-\theta$.

If $\theta \in \left(0, \frac{\pi}{2}\right)$, the corresponding fourth quadrant angle using $\left(-\frac{\pi}{2}, 0\right)$ is $-\theta$!

**Common error:** Confusing the fourth quadrant angle in standard position $-10^\circ$ with the $10^\circ$ angle that $-10^\circ$ makes with the positive $x$ axis.

**Practice:** (i) Find the first quadrant angle relatives of (a) $-80^\circ$ (b) $-5\pi/12$.

(ii) Find the fourth quadrant relatives of (a) $65^\circ$ (b) $\pi/3$.

**Answers:** (i)(a) $80^\circ$ (b) $5\pi/12$ (ii)(a) $-65^\circ$ (b) $-\pi/3$

**A Great Website for More Detail:** [oakroadsystems.com/twt/refangle.htm](http://oakroadsystems.com/twt/refangle.htm)

**MP3 Section VIII: Trigonometry 9**
9) Relating an Angle in Standard Position to its “Relatives” in the other Quadrants

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

For this question, use the following restrictions:

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Degrees: $0 &lt; \theta &lt; 90^\circ$ or Radians: $0 &lt; \theta &lt; \frac{\pi}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>Degrees: $90^\circ &lt; \theta &lt; 180^\circ$ or Radians: $\frac{\pi}{2} &lt; \theta &lt; \pi$</td>
</tr>
<tr>
<td>3</td>
<td>Degrees: $180^\circ &lt; \theta &lt; 270^\circ$ or Radians: $\pi &lt; \theta &lt; \frac{3\pi}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>Degrees: $-90^\circ &lt; \theta &lt; 0^\circ$ or Radians: $-\frac{\pi}{2} &lt; \theta &lt; 0$</td>
</tr>
</tbody>
</table>

**Problem:** State the standard position “relatives” of
(i) $50^\circ$  
(ii) $170^\circ$  
(iii) $250^\circ$  
(iv) $-60^\circ$ in each of the other quadrants.

**Solution:**
(i) Q2: $130^\circ$;  
Q3: $230^\circ$;  
Q4: $-50^\circ$  
(ii) Q1: $10^\circ$;  
Q3: $190^\circ$;  
Q4: $-10^\circ$  
(iii) Q1: $70^\circ$;  
Q2: $110^\circ$;  
Q4: $-70^\circ$  
(iv) Q1: $60^\circ$;  
Q2: $120^\circ$;  
Q3: $240^\circ$

**Note:** Remember that every angle has LOTS of names. Here, we were careful to specify which name we wanted to find.

**Common error:** You name the angle whose relatives you are trying to find using a range other than specified. For example, you might give $350^\circ$ when, for the questions on this page, the answer would be $-10^\circ$.

**Practice:** State the relatives of
(i) $\frac{\pi}{4}$  
(ii) $\frac{9\pi}{7}$  
(iii) $\frac{13\pi}{10}$  
(iv) $-\frac{\pi}{5}$.

**Answer:** Q ≡ Quadrant

(i) Q2: $\frac{3\pi}{4}$;  
Q3: $\frac{5\pi}{4}$;  
Q4: $-\frac{\pi}{4}$  
(ii) Q1: $\frac{2\pi}{7}$;  
Q2: $\frac{5\pi}{7}$;  
Q4: $-\frac{2\pi}{7}$

(iii) Q1: $\frac{3\pi}{10}$;  
Q2: $\frac{7\pi}{10}$;  
Q4: $-\frac{3\pi}{10}$  
(iv) Q1: $\frac{\pi}{5}$;  
Q2: $\frac{4\pi}{5}$;  
Q3: $\frac{6\pi}{5}$

**A Great Website for More Detail:** oakroadsystems.com/twt/refangle.htm
10) Trigonometric Ratios in Right Triangles: SOHCOHTOA

Problem:
From the triangle, identify all six trigonometric ratios for $\theta$.

Solution:

\[
\begin{align*}
\sin(\theta) &= \frac{O}{H} = \frac{4}{5} & \cos(\theta) &= \frac{A}{H} = \frac{3}{5} & \tan(\theta) &= \frac{O}{A} = \frac{4}{3} \\
\csc(\theta) &= \frac{1}{\sin(\theta)} = \frac{5}{4} & \sec(\theta) &= \frac{1}{\cos(\theta)} = \frac{5}{3} & \cot(\theta) &= \frac{1}{\tan(\theta)} = \frac{3}{4}
\end{align*}
\]

Note: These definitions only work if $0 < \theta < \frac{\pi}{2} = 90^\circ$. For other angles, we get the trig values from its “related” first quadrant angle and the CAST RULE.

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

Common error: Mixing up the opposite and adjacent sides.

Practice: From the triangle, identify all six trigonometric ratios for $\theta$.

\[
\begin{align*}
\sin(\theta) &= \frac{3}{5} & \cos(\theta) &= \frac{4}{5} & \tan(\theta) &= \frac{3}{4} & \csc(\theta) &= \frac{5}{4} & \sec(\theta) &= \frac{3}{4} & \cot(\theta) &= \frac{4}{3}
\end{align*}
\]

A Great Website for More Detail: [mathwords.com/s/sohcahtoa.htm](http://mathwords.com/s/sohcahtoa.htm)
11) Trigonometric Ratios Using the Circle: Part I

**Problem:** Find the sin, cos, and tan of an angle \( \theta \) which is **not** between 0° and 90°.

**Solution:** Draw the angle in standard position. (The illustrated \( \theta \) satisfies 90° < \( \theta \) < 180°.) It will **puncture** the unit circle, centre the origin, at a point \((x, y)\).

Then \( \sin(\theta) = y \), \( \cos(\theta) = x \), and \( \tan(\theta) = \frac{y}{x} \).

**Note:** If 0° < \( \theta \) < 90°, our picture would look like this:

Using the method of the solution above, \( \sin(\theta) = y \), \( \cos(\theta) = x \), and \( \tan(\theta) = \frac{y}{x} \). Using "triangle trig", \( \sin(\theta) = \frac{O}{H} = \frac{y}{1} = y \), \( \cos(\theta) = \frac{A}{H} = \frac{x}{1} = x \), and \( \tan(\theta) = \frac{O}{A} = \frac{y}{x} \). We get the same answers. Surprised? You shouldn't be. The method of the solution is just the LOGICAL extension of trig to angles not between 0° and 90°.

**Common error:** \( x \) and \( y \) will be either + or − depending on the quadrant where \( \theta \) punctures the circle. Getting the “signs” wrong (and therefore often the “sin’s” wrong!) is common!

**Practice:** If \( \theta \) punctures the circle in the fourth quadrant, what are the signs of \( x \) and \( y \) and how does this relate to the CAST RULE?

**Answer:** \( x \) is + and \( y \) is −. The sin and tan will be − while cos will be +.
This accounts for the C in the CAST RULE.

**A Great Website for more Detail:** themathpage.com/aTrig/unit-circle.htm#quad

**MP³ Section VIII: Trigonometry 12**
12) Trigonometric Ratios Using the Circle: Part II

**Problem:** In the diagram, $\theta$, where $90^\circ < \theta < 180^\circ$, is a second quadrant angle in standard position whose terminal side punctures the circle (centered at the origin and with radius 13) at the point P(−5,12). State the six trigonometric ratios of $\theta$.

**Solution:**

The second quadrant triangle with sides −5, 12, and 13 is the right triangle associated with the second quadrant angle $\theta$. Using this triangle with −5 as the adjacent side, 12 as the opposite side, and 13 as the hypotenuse, we have

\[
\sin(\theta) = \frac{O}{H} = \frac{12}{13} \quad \cos(\theta) = \frac{A}{H} = \frac{-5}{13} \quad \tan(\theta) = \frac{O}{A} = \frac{12}{-5} = -\frac{12}{5} \\
\csc(\theta) = \frac{H}{O} = \frac{13}{12} \quad \sec(\theta) = \frac{H}{A} = \frac{-13}{-5} = \frac{-13}{5} \quad \cot(\theta) = \frac{A}{O} = \frac{-5}{12}
\]

**Notes:** If $\alpha = \theta + 360^\circ$, then $\alpha$ will have the same second quadrant associated triangle as $\theta$ and so will have the same trig ratios. This is true for all angles of the form $\theta + 360k^\circ$, where $k \in \mathbb{Z}$. Note that of the three ratios sin, cos, and tan, only sin is positive. This is where the S in the CAST RULE (for the Sin in the second quadrant) comes from.

**Another Note:** We could have used the circle with radius 1 and the associated triangle with sides $\frac{12}{13}$ and $\frac{-5}{13}$. The resulting triangle is similar to the one we used above and so the trig ratios would be the same.

**And One More Note:** Every angle in standard position will have an “associated” right triangle. Take the end point $(a,b)$ on the terminal arm (the point where $\theta$, in standard position, punctures the circle) and draw a perpendicular to the $x$ axis. There is your triangle. The adjacent side is $a$, the opposite side is $b$, and the hypotenuse is $\sqrt{a^2 + b^2}$.

**Common error:** Mislabling the signs on the co-ordinates of the point where $\theta$ punctures the circle: for example, mislabeling P as (5,12) instead of (−5,12).

**Practice:** Suppose $\theta$, where $\pi < \theta < \frac{3\pi}{2}$, punctures the circle, centred at the origin and with radius 5, at the point (−4,−3). State the sin, cos, and tan of $\theta$.

(Hint: hypotenuse = 5)

**Answer:** $\sin(\theta) = -\frac{3}{5}$, $\cos(\theta) = -\frac{4}{5}$, $\tan(\theta) = -\frac{3}{4}$

**A Great Website for more Detail:** themathpage.com/aTrig/unit-circle.htm#quad

MP³ Section VIII: Trigonometry 13
13) Trigonometric Ratios for the $45^\circ$, $45^\circ$, $90^\circ$ Triangle

**Problem:** Find the values of all the six trigonometric ratios of $45^\circ = \frac{\pi}{4}$.

**Solution:** Let $\Delta ABC$ be an isosceles right triangle with $\angle B = 90^\circ$ and $AB = CB = 1$. Then $\angle A = \angle C = 45^\circ$ and $AC = \sqrt{2}$.

![Isosceles right triangle](image)

\[
\begin{align*}
\sin(45^\circ) &= \frac{O}{H} = \frac{1}{\sqrt{2}} & 
\cos(45^\circ) &= \frac{A}{H} = \frac{1}{\sqrt{2}} & 
\tan(45^\circ) &= \frac{O}{A} = 1 \\
\csc(45^\circ) &= \sqrt{2} & 
\sec(45^\circ) &= \sqrt{2} & 
\cot(45^\circ) &= 1
\end{align*}
\]

**Note:** We set $AB = CB = 1$. If we had set $AB = CB = 3$, then we would have $AC = 3\sqrt{2}$. Then, $\sin(45^\circ) = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$, which is the same as before. In other words, the sides stay in proportion to one another (similar triangles!) and the ratios are unchanged.

**Common error:** Determining the hypotenuse $AC = 2$.

**Practice:** Find the values of the six trigonometric ratios of the second quadrant angle $135^\circ$. (Hint: $135^\circ$ has related first quadrant angle $180^\circ - 135^\circ = 45^\circ$. Use the ratios for $45^\circ$ and the CAST RULE.)

**Answer:**
\[
\begin{align*}
\sin(135^\circ) &= \frac{1}{\sqrt{2}} & 
\cos(135^\circ) &= -\frac{1}{\sqrt{2}} & 
\tan(135^\circ) &= -1 \\
\csc(135^\circ) &= \sqrt{2} & 
\sec(135^\circ) &= -\sqrt{2} & 
\cot(135^\circ) &= -1
\end{align*}
\]

**A Great Website for more Detail:**
[hyperad.com/tutoring/math/trig/Trigonometric%20Functions%20of%20Common%20Angles.html](https://hyperad.com/tutoring/math/trig/Trigonometric%20Functions%20of%20Common%20Angles.html)
14) Trigonometric Ratios for the 30°, 60°, 90° Triangle

**Problem:** Find the values of all the six trigonometric ratios of \( 60° = \frac{\pi}{3} \) and \( 30° = \frac{\pi}{6} \).

**Solution:** Let \( \triangle ABC \) be an equilateral triangle with \( AB = CB = AC = 2 \). Then \( \angle A = \angle B = \angle C = 60° \). Drop a perpendicular from \( A \) to meet \( BC \) at \( D \). \( \therefore \triangle ADB \cong \triangle ADC \) and so \( \angle BAD = \angle CAD = 30° \) and \( BD = CD = 1 \).

Finally, using Pythagoras, \( AD = \sqrt{2^2 - 1^2} = \sqrt{3} \).

\[
\begin{align*}
\sin(60°) &= \frac{O}{H} = \frac{\sqrt{3}}{2} & \cos(60°) &= \frac{A}{H} = \frac{1}{2} & \tan(60°) &= \frac{O}{A} = \sqrt{3} \\
csc(60°) &= \frac{2}{\sqrt{3}} & \sec(60°) &= 2 & \cot(60°) &= \frac{1}{\sqrt{3}} \\
\sin(30°) &= \frac{O}{H} = \frac{1}{2} & \cos(30°) &= \frac{A}{H} = \frac{\sqrt{3}}{2} & \tan(30°) &= \frac{O}{A} = \frac{1}{\sqrt{3}} \\
csc(30°) &= 2 & \sec(30°) &= \frac{2}{\sqrt{3}} & \cot(30°) &= \sqrt{3}
\end{align*}
\]

**Note:** The adjacent side for 60°, 1, is the opposite side for 30°. The opposite side for 60°, \( \sqrt{3} \), is the adjacent side for 30°. This always happens with “complementary” angles. The adjacent for \( \theta \) is the opposite for \( 90° - \theta \).

**Common error:** Forgetting where to put the 1 and the \( \sqrt{3} \). Everybody remembers that 2 is the hypotenuse.

**Practice:** Find the sin, cos and tan of the third quadrant angle 210°. (Hint: 210° has related first quadrant angle 210 – 180 = 30°. Use the ratios for 30° and the CAST RULE.)

**Answer:**
\[
\begin{align*}
\sin(210°) &= -\frac{1}{2} & \cos(210°) &= -\frac{\sqrt{3}}{2} & \tan(210°) &= \frac{1}{\sqrt{3}}
\end{align*}
\]

A Great Website for more Detail:

[hyperad.com/tutoring/math/trig/Trigonometric%20Functions%20of%20Common%20Angles.html](hyperad.com/tutoring/math/trig/Trigonometric%20Functions%20of%20Common%20Angles.html)
15) Trigonometric Ratios for the 0°, 90°, 180°, 270°; ie., For Angles ON the Axes

First PLEASE re-read “11) Trigonometric Ratios Using the Circle: Part I”.

Problem: Find the six trig ratios for 0° = 0 radians and 90° = \( \frac{\pi}{2} \) radians.

Solution: \( \theta = 0^\circ \) punctures the unit circle at (1,0) and 90° punctures this circle at (0,1).

\[
\begin{array}{c|c|c|c}
\theta & \sin(\theta) & \cos(\theta) & \tan(\theta) \\
\hline
0^\circ & 0 & 1 & 0 \\
& \text{csc}(0^\circ) \text{ is undefined} & \sec(0^\circ) = 1 & \cot(0^\circ) \text{ is undefined} \\
\end{array}
\]

90°

\[
\begin{array}{c|c|c|c}
\theta & \sin(\theta) & \cos(\theta) & \tan(\theta) \\
\hline
90^\circ & 1 & 0 & \text{undefined} \\
& \csc(90^\circ) = 1 & \sec(90^\circ) \text{ is undefined} & \cot(90^\circ) = 0 \\
\end{array}
\]

Notes: Division by zero is undefined and that is why there are undefined trig ratios for angles that puncture the axes, that is, where one of the co-ordinates is 0. Also, the ratios for 360\(k\), \(k \in \mathbb{Z}\), are the same as for 0°; the ratios for 90°+360\(k\), \(k \in \mathbb{Z}\), are the same as for 90°.

Common error: \( \sin(0^\circ) = 1 \), \( \cos(0^\circ) = 0 \), etc.

Practice: Find the sin, cos and tan of \( \frac{7\pi}{2} \) (i) \( \frac{7\pi}{2} \) (ii) \( -3\pi \)

Hint: \( \frac{7\pi}{2} \) is co-terminal with \( -\frac{\pi}{2} \) and punctures the circle at (0,−1); \( -3\pi \) is co-terminal with \( \pi \) and punctures the circle at (−1,0).

Answer: (i) \( \sin\left(\frac{7\pi}{2}\right) = -1 \) \( \cos\left(\frac{7\pi}{2}\right) = 0 \) \( \tan\left(\frac{7\pi}{2}\right) \) is undefined

(ii) \( \sin(-3\pi) = 0 \) \( \cos(-3\pi) = -1 \) \( \tan(-3\pi) = 0 \)

A Great Website for More Detail:

[jwbales.home.mindspring.com/precal/part4/part4.2.html](jwbales.home.mindspring.com/precal/part4/part4.2.html)
16) CAST RULE

Problems: 1) Given \( \tan(\theta) = -\frac{4}{3} \), find the values of \( \sin(\theta) \) and \( \cos(\theta) \) if
(i) \( \theta \) is a second quadrant angle.
(ii) \( \theta \) is a fourth quadrant angle.

2) Why can’t \( \theta \) be a first or third quadrant angle?

Solutions: (i) If \( \tan(\theta) = -\frac{4}{3} \), and \( \theta \) is a second quadrant angle, then \( x < 0 \) and \( y > 0 \). Draw \( \theta \) with terminal point \((-3,4)\) and drop a perpendicular to the \(x\) axis to get the
“associated” right triangle. (In the picture, we assumed \( \frac{\pi}{2} < \theta < \pi \), but \( \theta \) could have satisfied, for example, \(-\frac{3\pi}{2} < \theta < -\pi \).) Then \( \sin(\theta) = \frac{4}{5} \) and \( \cos(\theta) = -\frac{3}{5} \).

(ii) If \( \tan(\theta) = -\frac{4}{3} \), and \( \theta \) is a fourth quadrant angle, then \( x > 0 \) and \( y < 0 \). Draw \( \theta \) with terminal point \((3,-4)\) and drop a perpendicular to the \(x\) axis to get the “associated”
right triangle. Then \( \sin(\theta) = -\frac{4}{5} \) and \( \cos(\theta) = \frac{3}{5} \).

2) \( \tan \) is positive in quadrants 1 and 3 so we can’t have \( \tan(\theta) = -\frac{4}{3} \).

Note:

<table>
<thead>
<tr>
<th>Quadrant 1</th>
<th>Quadrant 2</th>
<th>Quadrant 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos )</td>
<td>( \sin )</td>
<td>( \tan )</td>
</tr>
</tbody>
</table>

Common error: Given \( \tan(\theta) = -\frac{4}{3} \) in the second quadrant, setting \( x = 3 \) and \( y = -4 \).

Practice: Given \( \cos(\theta) = -\frac{2}{3} \) and \( \theta \) is a third quadrant angle, find \( \sin(\theta) \) and \( \tan(\theta) \).

Answer: \( \sin(\theta) = -\frac{\sqrt{5}}{3} \), \( \tan(\theta) = \frac{\sqrt{5}}{2} \)

A Great Website for More Detail:

worsleyschool.net/science/files/cast/castdiagram.html

MP³ Section VIII: Trigonometry 17
17) Sine Law: Find an Angle

Problem: Find $\angle C$ and $\angle A$.

Solution: By the Sine Law,

By the Sin Law, \[
\frac{\sin(C)}{c} = \frac{\sin(B)}{b}
\]

\[\frac{b=10\sqrt{3}}{c=10\sqrt{2}}\]

\[\sin(60^\circ) = \frac{\sqrt{3}}{2}\]

\[\therefore \sin(C) = \frac{c \cdot \sin(B)}{b} = \frac{10\sqrt{2}}{10\sqrt{3} \times \frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{2}}.
\]

\[\therefore \angle C = 45^\circ \text{ and } \angle A = 180^\circ - (60^\circ + 45^\circ) = 75^\circ
\]

Notes: $\sin(C) = \frac{1}{\sqrt{2}} \Rightarrow C = 45^\circ$ or $135^\circ$. (Sin is + in both the first and second quadrants!)

The angles in $\triangle ABC$ must sum to $180^\circ$ and so $C = 135^\circ$ is too big. However, sometimes, there are two solutions!

Another Note: This works when you have two sides and an angle which is not the contained angle. If you have the contained angle, you need the Cosine Law.

Common error: $a \sin(A) = b \sin(B)$

Practice: In $\triangle ABC$, $c = AB = 10$, $b = AC = 12$ and $\angle B = 60^\circ$. Find $\angle C$ and $\angle A$ accurate to one decimal place. (Hint: to solve $\sin(C) = x$ on most calculators, use “x 2nd Function sin”.)

Answers: $\angle C \approx 46.2$ $\angle A \approx 73.8$

A Great Website for More Detail:

ilearn.senecac.on.ca/learningobjects/MathConcepts/SineLaw/main.htm

MP³ Section VIII: Trigonometry 18
18) Sine Law: Find a Side

Problem: Find (i) the exact value of $a$ and (ii) $c$ accurate to two decimals.

Solution: $\angle C = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$

By the Sine Law:

$$a = \frac{b \sin A}{\sin B} = \frac{20 \sqrt{3} \times 2}{\sqrt{3}} = \frac{1}{\sqrt{2}} \times 40 = 20 \sqrt{2}$$

$$c = \frac{b \sin C}{\sin B} = \frac{20 \sqrt{3} \times 2}{\sqrt{3}} = 0.9659 \times 40 \approx 38.64$$

Note: The Sine Law contains four quantities: two angles in a triangle and the two sides opposite these angles. The Sine Law is useful when you have three of these four quantities.

Common error: $a \sin(A) = b \sin(B)$

Practice: In $\triangle ABC$, $\angle A = 60^\circ$, $\angle B = 45^\circ$, and side $b = AC = 36$. Find the lengths of sides $a = BC$ and $c = AB$ accurate to two decimal places.

Answer: $a \approx 44.10 \quad c \approx 49.18$

A Great Website for More Detail:

ilearn.senecac.on.ca/learningobjects/MathConcepts/SineLaw/main.htm
19) **Cosine Law: Find an angle**

**Problem:** In $\triangle ABC$, use the Cosine Law to find $\angle B$ to the nearest degree.

**Solution:** Using the Cosine Law:

$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac} = \frac{49 + 100 - 81}{2(7)(10)} = \frac{68}{140} = \frac{17}{35}$$

Make sure your calculator is in “DEGREE” mode!

\[ \therefore B \approx 60.94^\circ \]

**Note:** Make sure your calculator is in degree mode. To find an angle using the Cosine Law you need to know the lengths of all the three sides of the triangle.

**Common error:** Using “RADIAN” mode when the answer is required in degrees or vice-versa. The answer here in radians is 1.06. Remember that 1 radian is about 57°. So, 1.06 radians is about 61°.

**Practice:** In $\triangle ABC$, we have $a = BC = 2$, $b = AC = 3$ and $c = AB = 4$. Find $\angle C$ to the nearest degree.

\[ \therefore C = 105^\circ \]

**A Great Website for More Detail:**

ilearn.senecac.on.ca/learningobjects/MathConcepts/CosineLaw/main.htm

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**MP³ Section VIII: Trigonometry 20**
20) Cosine Law: Find a Side

**Problem:** In \( \Delta ABC \), use the Cosine Law to find \( c = AB \) correct to two decimal places.

**Solution:** Using the Cosine Law:

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

\[
\begin{align*}
c^2 &= 71 + 22^2 - 2 \cdot 7 \cdot 22 \cdot \cos 60^\circ \\
&= 49 + 100 - 140(0.5) = 79
\end{align*}
\]

Make sure your calculator is in “DEGREE” mode!

\[c \approx 8.89\]

**Note:** To find a side using the Cosine Law, you need two sides and the contained angle. If you have a “non-contained” angle, use the Sine Law to find the contained angle and then use the Cosine Law or the Sine Law to find the required side.

**Common error:** \( c^2 = a^2 + b^2 + 2ab \cos C \)

**Practice:** In \( \Delta ABC \), we have \( b = AC = 6, c = AB = 4 \) and \( \angle A = 35^\circ \). Find \( a = BC \), correct to two decimals.

**Answer:** \( a \approx 3.56 \)

**A Great Website for More Detail:**

[ilearn.senecac.on.ca/learningobjects/MathConcepts/CosineLaw/main.htm](ilearn.senecac.on.ca/learningobjects/MathConcepts/CosineLaw/main.htm)
21) The Graphs of the Sin, Cos, and Tan Functions

Problem: Graph for $0 \leq \theta \leq 2\pi$ : (i) $y = \sin(\theta)$  (ii) $y = \cos(\theta)$  (iii) $y = \tan(\theta)$

Solution: (i) $y = \sin(\theta)$  (ii) $y = \cos(\theta)$  (iii) $y = \tan(x)$

Note: The range of $y = \sin(\theta)$ and $y = \cos(\theta)$ is $[-1,1]$. The range of $y = \tan(\theta)$ is $\mathbb{R}$.

Common error: Confusing, for example, the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, which is on the graph of both $y = \sin(\theta)$ and $y = \cos(\theta)$, with the terminal point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ of $\theta = \frac{\pi}{4}$, on the circle $x^2 + y^2 = 1$. On this circle, for each angle $\theta$, we puncture the circle in a point $(x, y)$ and we define $x = \cos(\theta)$ and $y = \sin(\theta)$. The values of cos and sin lead to the graphs above.

Practice: Name all values of $\theta$ where (i) $\sin(\theta) = 0$  (ii) $\cos(\theta) = 0$  (iii) $\tan(\theta) = 0$

Answer: (i) $k\pi, k \in \mathbb{Z}$  (ii) $\frac{\pi}{2} + k\pi, k \in \mathbb{Z}$  (iii) $k\pi, k \in \mathbb{Z}$

A Great Website for More Detail:

library.thinkquest.org/20991/alg2/trig.html#graph
22) Period of the Sin, Cos, and Tan Functions

Problems: State the period of each of the following:

1) (i) \( y = \sin(x) \)  (ii) \( y = \sin(2x) \)  (iii) \( y = \sin\left(\frac{x}{2}\right) \)

2) (i) \( y = \cos(x) \)  (ii) \( y = \cos(3x) \)  (iii) \( y = \cos\left(\frac{x}{3}\right) \)

3) (i) \( y = \tan(x) \)  (ii) \( y = \tan(4x) \)  (iii) \( y = \tan\left(\frac{\pi}{2}x\right) \)

Solutions: 1)(i) \( 2\pi \)  (ii) \( \frac{2\pi}{2} = \pi \)  (iii) \( \frac{2\pi}{1/2} = 4\pi \)

2)(i) \( 2\pi \)  (ii) \( \frac{2\pi}{3} \)  (iii) \( \frac{2\pi}{1/3} = 6\pi \)

3)(i) \( \pi \)  (ii) \( \frac{\pi}{4} \)  (iii) \( \frac{\pi}{\pi/2} = 2 \)

Note: If the function \( y = f(x) \) has period \( P \), then \( y = f(kx) \) has period \( \frac{P}{k} \).

Common error: The period of \( y = \tan(2x) \) is \( \frac{2\pi}{2} = \pi \).

Practice: Because the period of \( y = \sin(x) \) is \( 2\pi \), we know that for any integer \( k \), \( \sin(x + 2k\pi) = \) _______

Answer: \( \sin(x) \)

A Great Website for More Detail: mathsrevision.net/alevel/pages.php?page=36

MP³ Section VIII: Trigonometry 23
23) The Graphs of the Csc, Sec, and Cot Functions

**Problem:** Graph for $0 \leq \theta \leq 2\pi$: (i) $y = \csc(\theta)$ (ii) $y = \sec(\theta)$ (iii) $y = \cot(\theta)$

**Solution:**

(i) $y = \csc(\theta)$  

(ii) $y = \sec(\theta)$  

(iii) $y = \cot(\theta)$

![Graphs of Csc, Sec, and Cot Functions](image)

**Note:** The range of $y = \csc(\theta)$ and $y = \sec(\theta)$ is $(-\infty, -1] \cup [1, \infty)$. The range of $y = \cot(\theta)$ is $\mathbb{R}$.

**Common error:** Completely misunderstanding why the range of csc is from +1 up and from −1 down: when $\sin(\theta)$ is inside $(-1,1)$, then $\frac{1}{\sin(\theta)} = \csc(\theta)$ is outside $(-1,1)$. When $\sin(\theta) = \pm 1$, so does $\csc(\theta)$.

**Practice:** Name all values of $\theta$ where (i) $\csc(\theta)$ (ii) $\sec(\theta)$ (iii) $\cot(\theta)$ is undefined.

**Answer:** (i) $k\pi$, $k \in \mathbb{Z}$  
(ii) $\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$  
(iii) $k\pi$, $k \in \mathbb{Z}$

**A Great Website for More Detail:**

[regentsprep.org/Regents/math/algtrig/ATT7/othergraphs.htm](http://regentsprep.org/Regents/math/algtrig/ATT7/othergraphs.htm)
24) Trig Formulas That You Should Know

**Problem:** Complete the following formulas:

(i) $\sin^2(\theta) + \cos^2(\theta) =$  \hfill \square \hfill ; \hfill 1 + \tan^2(\theta) =$ \hfill \square \hfill 

(ii) $\sin(-\theta) =$ \hfill \square \hfill ; \hfill \cos(-\theta) =$ \hfill \square \hfill 

(iii) $\sin\left(\frac{\pi}{2} - \theta\right) =$ \hfill \square \hfill ; \hfill \cos\left(\frac{\pi}{2} - \theta\right) =$ \hfill \square \hfill 

(iv) $\sin(A \pm B) =$ \hfill \square \hfill ; \hfill \cos(A \pm B) =$ \hfill \square \hfill 

(v) $\sin(2A) =$ \hfill \square \hfill ; \hfill \cos(2A) =$ \hfill \square \hfill 

**Solution:**

(i) $\sin^2(\theta) + \cos^2(\theta) = \underline{1}$; \hfill $1 + \tan^2(\theta) =$ \hfill $\sec^2(\theta)$ \hfill 

(ii) $\sin(-\theta) = -\sin(\theta)$; \hfill $\cos(-\theta) =$ \hfill $\cos(\theta)$ \hfill 

(iii) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$; \hfill $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ \hfill 

(iv) $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$; \hfill 

(v) $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$ \hfill 

(vi) $\sin(2A) = 2\sin(A)\cos(A)$; \hfill $\cos(2A) =$ \hfill $\cos^2(A) - \sin^2(A)$ \hfill 

**Note:** These formulas are used over and over and over again. Learn them.

**Common error:** $\cos(A \pm B) = \cos(A)\cos(B) \pm \sin(A)\sin(B)$

**Practice:** Complete the formulas:

(i) $\cot^2(\theta) + 1 =$ \hfill \square \hfill ; \hfill (ii) $\tan(-\theta) =$ \hfill \square \hfill ; \hfill (iii) $\tan\left(\frac{\pi}{2} - \theta\right) =$ \hfill \square \hfill 

(iv) $\tan(A \pm B) =$ \hfill \square \hfill ; \hfill (v) $\tan(2A) =$ \hfill \square \hfill 

**Answer:**

(i) $\csc^2(\theta)$ \hfill (ii) $-\tan(\theta)$ \hfill (iii) $\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$ \hfill 

(iv) $\tan(A \pm B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$ \hfill (v) $\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$ \hfill 

**A Great Website for More Detail:**

[analyzemath.com/trigonometry/trigonometric_formulas.html](http://analyzemath.com/trigonometry/trigonometric_formulas.html)
Solutions Part IX
LOGS
AND
MATH EXPONENTS!

w \equiv \{\sqrt{\text{❤}}\}^2

math

MP^3 Section IX: Exponents and Logarithms 1
1) Exponents

Problems: 1) Evaluate: (i) \(2^3\) (ii) \(\left(\frac{3}{5}\right)^3\) (iii) \(4^{-2}\) (iv) \(10^0\) (v) \(\left(\frac{1}{0.01}\right)^{-3}\)

2) Simplify: (i) \(\frac{x^5x^4}{x^3}\) (ii) \(w^{-1}\) (iii) \(\frac{1}{w^3}\) (iv) \(\left(z^3\right)^{10}\) (v) \(\left(\frac{a^2b^3}{c^2}\right)^5\)

Solution: 1) (i) \(2^3 = 2 \times 2 \times 2 = 8\) (ii) \(\left(\frac{3}{5}\right)^3 = \frac{27}{125}\) (iii) \(4^{-2} = \frac{1}{4^2} = \frac{1}{16}\)

(iv) \(10^0 = 1\) (v) \(\left(\frac{1}{0.01}\right)^{-3} = (0.01)^3 = (10^{-2})^3 = 10^{-6} = \frac{1}{1000000}\)

2) (i) \(\frac{x^5x^4}{x^7} = x^{5+4-7} = x^2\) (ii) \(w^{-1} = \frac{1}{w}\) (iii) \(\frac{1}{w^3} = w^3\) (iv) \(\left(z^{2/3}\right)^{10} = z^{20/3}\) (v) \(\left(\frac{a^2b^3}{c^2}\right)^5 = a^{10}b^{15}\)

Notes: Multiplication is a short form for repeated addition—\(7 + 7 + 7 + 7 = 7 \times 4\)
Exponentiation is a short form for repeated multiplication—\(7 \times 7 \times 7 \times 7 = 7^4\)

Common error: \(\left(3^5\right)^4 = 3^9\), i.e., adding exponents when you should be multiplying.

Practice: 1) Evaluate: (i) \(2^5\) (ii) \(\left(\frac{2}{3}\right)^{-2}\) (iii) \(10^0\) (iv) \(0^0\)

2) Simplify: (i) \(\frac{x^5y^2}{x^{11}y^5}\) (ii) \(h^{-1}\) (iii) \(\frac{1}{h^{-1}}\) (iv) \(\left(\frac{ab^{-2}}{c^{-3}}\right)^5\)

Answers: 1) (i) 32 (ii) \(\frac{9}{4}\) (iii) 1 (iv) Does not exist. “0^0” is undefined.

2) (i) \(\frac{y^7}{x^6}\) (ii) \(\frac{1}{h}\) (iii) \(h\) (iv) \(\left(\frac{ab^{-2}}{c^{-3}}\right)^5 = \frac{a^5c^{15}}{b^{10}}\)

A Great Website for More Detail: purplemath.com/modules/exponent.htm

MP³ Section IX: Exponents and Logarithms 2
2) Logarithms (Log means FIND THE EXPONENT!)

**Problems:** 1) Evaluate: (i) \( \log_2 8 \) (ii) \( \log_2 \left( \frac{1}{8} \right) \) (iii) \( \log_3 1 \) (iv) \( \log_5 5 \)

(v) \( \log_{10} 10000 \) (vi) \( \ln(7) \) *

2) Expand using log properties: \( \ln \left( \frac{x^3 y^{1/2}}{z^4} \right) \)

3) Change \( \log_7 \) to log with base 3, then with base 10, and finally with base \( e \).

*“e” and “ln” refer to the “natural logarithm”. If you have not taken calculus, you may be totally unfamiliar with e. If so, treat it as a constant just as you would, for example, the letter \( a \).*

**Solutions:** 1) (i) \( \log_2 8 = 3 \) (ii) \( \log_2 \left( \frac{1}{8} \right) = -3 \) (iii) \( \log_3 1 = 0 \) (iv) \( \log_5 5 = 1 \)

(v) \( \log_{10} 10000 = 4 \) (vi) \( \ln(7) = 7 \)

2) \( \ln \left( \frac{x^3 y^{1/2}}{z^4} \right) = \ln(x^3) + \ln(y^{1/2}) - \ln(z^4) = 3 \ln x + \frac{1}{2} \ln y - 4 \ln z \)

3) \( \log_3 7 = \frac{\log_5 7}{\log_5 3} = \frac{\log 7}{\log 5} = \frac{\ln 7}{\ln 5} \)

**Notes:** “Log” means “Find the exponent!”.
“log” with no base is short for \( \log_{10} \) and \( \ln \) is short for \( \log_e \).

**Common error:** Many students confuse \( (\log_2 8)^3 = \log_2 8 \times \log_2 8 \times \log_2 8 = 3^3 = 27 \)
with \( \log_2 (8^3) = \log_2 (8 \times 8 \times 8) = \log_2 (2^9) = 9 \).

**Practice:** 1) Evaluate: (i) \( \log_5 81 \) (ii) \( \log_5 \left( \frac{1}{125} \right) \) (iii) \( \log_{0.1} 1 \) (iv) \( \log_2 0 \)

(v) \( \log_{10000} \frac{1}{10} \) (vi) \( \ln \left( \frac{1}{e^7} \right) \)

2) Expand using log properties: \( \ln \left( x^3 + y^{1/2} - z^4 \right) \)

3) Change \( \log_7 7 \) to log with base 7.

**Answers:** 1) Evaluate: (i) \( 4 \) (ii) \( -3 \) (iii) \( 0 \) (iv) does not exist (v) \( -4 \) (vi) \( -7 \)

2) \( \ln \left( x^3 + y^{1/2} - z^4 \right) \). You can’t expand this at all.

3) \( \frac{1}{\log_7 5} \).

A Great Website for More Detail: purplemath.com/modules/logrules.htm

MP³ Section IX: Exponents and Logarithms 3
3) Exponential Graphs

**Problem:** (i) Graph the exponential functions \( y = 2^x \) and \( y = 3^x \) on the same set of axes.
(ii) Graph the exponential functions \( y = 2^{-x} = \frac{1}{2^x} \) and \( y = 3^{-x} = \frac{1}{3^x} \) on the same set of axes.

**Solution:**

(i) Because the base is bigger (and greater than 1), \( y = 3^x \) goes up faster than \( y = 2^x \) when \( x > 0 \) and \( y \) approaches 0 faster when \( x \to -\infty \). Also, \( y = a^x \) is always +!

(ii) \( y = a^{-x} \) and \( y = a^x \) are mirror images in the \( y \) axis.

**Notes:**

- \( y = 3^x \) goes up faster than \( y = 2^x \) when \( x > 0 \) and \( y \) approaches 0 faster when \( x \to -\infty \).
- Also, \( y = a^x \) is always +!

**Common error:** Drawing the graph of \( y = 2^x \) so that it appears to cross the \( x \) axis when \( x \to -\infty \).

**Practice:** Graph \( y = 10^x \) and \( y = 10^{-x} \) on the same set of axes.

**Answer:**

A Great Website for More Detail: [purplemath.com/modules/graphexp.htm](http://purplemath.com/modules/graphexp.htm)
4) Logarithmic Graphs

**Problem:** Graph the functions $y = \log_2(x)$ and $y = \log_3(x)$ on the same set of axes.

**Solution:**

![Graph of log_2(x) and log_3(x)](image)

**Notes:** Because the base is bigger (and greater than 1), $y = \log_3(x)$ goes up more slowly than $y = \log_2(x)$ when $x > 1$ and $y$ approaches $-\infty$ faster when $x \to 0^+$. The domain of $y = \log_a(x)$ is $(0, \infty)$!

**Common error:** Drawing the graph of $y = \log_2(x)$ so that it appears to cross the $y$ axis when $x \to 0^+$. Even worse, this means you are allowing $x < 0$ into $y = \log_2(x)$. NO!

**Practice:** Graph $y = \log_2(x)$ and $y = \log_2\left(\frac{1}{x}\right)$ on the same set of axes.

**Hint:** $\log_2\left(\frac{1}{x}\right) = \log_2\left(x^{-1}\right) = -\log_2(x)$

**Answer:**

![Graph of log_2(x) and 1/x)](image)

A Great Website for More Detail: [purplemath.com/modules/graphlog.htm](http://purplemath.com/modules/graphlog.htm)

MP³ Section IX: Exponents and Logarithms 5
5) Exponents to Logarithms and Vice-Versa

**Problems:**
1) Change to a log equation: (i) $32=2^5$ (ii) $y=10^x$ (iii) $y=4x^k$ (Use base 10.)
2) Change to an exponential equation: (i) $\log_{3}81=4$ (ii) $y=\log_{5}x$

**Solutions:**
1)(i) $32=2^5 \iff 5 = \log_{2}32$  (ii) $y=10^x \iff x = \log_{10}(y)=\log(y)$  
   (iii) $y=4x^k \iff \log(y) = \log(4x^k) = \log(4) + k\log(x)$
2)(i) $\log_{3}81=4 \iff 81 = 3^4$  (ii) $y=\log_{5}x \iff x = 5^y$

**Note:** $y = a^x \iff x = \log_a(y)$

**Common error:** $y = 4x^k \iff k = \log_{4k}(y)$

**Practice:**
1) Change to a log equation: (i) $\frac{8}{27} = \left(\frac{2}{3}\right)^3$ (ii) $y = 0.5x^{k-1}$ (Use base 10.)
2) Change to an exponential equation: (i) $\log(0.001) = -3$  (ii) $y + 2 = \log_{5}(3x - 1)$

**Answers:**
1)(i)$\log_{2/3}\left(\frac{8}{27}\right) = 3$  (ii) $\log(y) = \log(.5) + (k-1)\log(x)$
2)(i) $10^{-3} = .001$  (ii) $5^{x+2} = 3x - 1$

**A Great Website for More Detail:** [purplemath.com/modules/logs.htm](http://purplemath.com/modules/logs.htm)
6) Using a Calculator to Evaluate Exponents and Logs

Problems: Use a calculator to give answers rounded to two decimal places.

1) (i) $2^{3.3}$  (ii) $5^{1/7}$  (iii) $(-10)^{1/3}$
2) (i) $\log(25)$  (ii) $\ln(25)$  (iii) $\log_3(25)$

*“e” and “ln” refer to the “natural logarithm”. If you have not taken calculus, you may be totally unfamiliar with e. If so, treat it as a constant just as you would, for example, the letter a.*

Solutions:  

1)(i) $2^{3.3} = 9.85$  (ii) $\sqrt[7]{5} = 5^{1/7} = 1.26$  (iii) $(-10)^{1/3}$. Most calculators won't accept a base < 0. We know the answer should be negative. So 

$$(-10)^{1/3} = -\sqrt[3]{10} = -2.15$$

2)(i) $\log(25) = 1.40$  (ii) $\ln(25) = 3.22$  (iii) $\log_3(25) = 4.64$ OR = 4.64

Note: Use the change of base formula when given a base other than 10 or $e$:

$$\log_a(b) = \frac{\log(b)}{\log(a)} = \frac{\ln(b)}{\ln(a)}$$

Common error: Believing your calculator when it answers INCORRECTLY when you enter $-2 \times 3$. Most calculators will report, “error” because they are not programmed to handle a negative base. Of course, $(-2)^3 = -8$.

Practice: Use a calculator to give answers rounded to two decimal places.

1)(i) $5^{3.1}$  (ii) $2^{1/5}$  (iii) $(-1001)^{1/3}$
2)(i) $\log(6)$  (ii) $\ln(6)$  (iii) $\log_7(6)$

Answers:  1)(i) 146.83  (ii) 1.15  (iii) $-10.00$  2)(i) .78  (ii) 1.79  (iii) .92

A Great Website for More Detail:

geocities.com/Athens/Thebes/5118/scicalc/scicalc.htm